

A II_1 FACTOR APPROACH TO THE KADISON-SINGER PROBLEM

SORIN POPA

Dedicated to R.V. Kadison and I.M. Singer

ABSTRACT. We show that the Kadison-Singer problem, asking whether the pure states of the diagonal subalgebra $\ell^\infty\mathbb{N} \subset \mathcal{B}(\ell^2\mathbb{N})$ have unique state extensions to $\mathcal{B}(\ell^2\mathbb{N})$, is equivalent to a similar statement in II_1 factor framework, concerning the ultrapower inclusion $D^\omega \subset R^\omega$, where D is the Cartan subalgebra of the hyperfinite II_1 factor R , and ω is a free ultrafilter. While we do not settle the problem in this latter form, we prove that if A is any singular maximal abelian subalgebra of R , then the inclusion $A^\omega \subset R^\omega$ does satisfy the Kadison-Singer property.

0. INTRODUCTION

A famous problem posed by Kadison and Singer in the late 1950s ([KS]) asks whether any pure state on the diagonal $\ell^\infty\mathbb{N}$ of the algebra $\mathcal{B}(\ell^2\mathbb{N})$, of all linear bounded operators on the Hilbert space $\ell^2\mathbb{N}$, has unique state extension to $\mathcal{B}(\ell^2\mathbb{N})$. We will refer to this property of the inclusion of algebras $\ell^\infty\mathbb{N} \subset \mathcal{B}(\ell^\infty\mathbb{N})$ as the *Kadison-Singer property*. As already pointed out in [KS], it is equivalent to the following property for operators on the Hilbert space, known as the *paving property*: if $x \in \mathcal{B}(\ell^2\mathbb{N})$ has only 0 on the diagonal, then for any $\varepsilon > 0$, there exists a finite partition of \mathbb{N} into subsets Y_1, \dots, Y_n , such that if $p_i \in \ell^\infty\mathbb{N}$ denotes the characteristic function of Y_i , viewed as a diagonal operator on $\ell^2\mathbb{N}$, then $\|\sum_{i=1}^n p_i x p_i\| \leq \varepsilon \|x\|$. It was later shown in [An1, An2] that this is in fact equivalent to the following finite dimensional version of the property, known as the *uniform paving property*: for any $\varepsilon > 0$, there exists $n = n(\varepsilon)$ such that for any m and any $x \in \mathcal{B}(\ell_m^2)$ with 0 on the diagonal, there exists a partition of $\{1, 2, \dots, m\}$ into n sets Y_i , such that the corresponding diagonal operators p_i satisfy $\|\sum_i p_i x p_i\| \leq \varepsilon \|x\|$.

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The Kadison-Singer problem has attracted much interest over the years, proving to have deep connections to a large number of fields of mathematics, with many interesting equivalent re-formulations. Several partial results have been obtained so far (see e.g. [A1], [A2], [AkA], [BT], [BeHKW], etc), showing for instance that certain classes of operators in $\mathcal{B}(\ell^2\mathbb{N})$ can indeed be paved. We refer the reader to [CaFTW] for a beautiful, comprehensive account on the state of the art on this problem, and on its interdisciplinary aspects.

In this paper, we attempt a new approach to the problem, based on a reformulation in II_1 factor framework. Recall that a II_1 *factor* is a von Neumann algebra M that is infinite dimensional, has trivial center and a completely additive trace state τ . II_1 factors can be viewed as non-commutative versions of the function algebra $L^\infty([0, 1])$, with the rôle of the Lebesgue integral $\int \cdot d\mu$ played by the positive functional $\tau : M \rightarrow \mathbb{C}$ which satisfies $\tau(1) = 1$ (it is a *state*) and $\tau(xy) = \tau(yx)$, $\forall x, y \in M$ (it is a *trace*). A specific type of (non-commutative) analysis has been developed in this framework, often exploiting the interplay between the operator norm and the Hilbert-norm implemented by the trace, as well as ergodicity properties of the Ad -action of the unitary group of M . One should note that the algebra $M_{m \times m}(\mathbb{C})$, of m by m matrices with complex entries ($\simeq \mathcal{B}(\ell_m^2)$), has both a trace state (given by the normalized trace tr) and is a factor, but it is finite dimensional. However, inductive limits and ultraproducts of these algebras give rise to II_1 factors.

Thus, the most “basic” example of a II_1 factor is the *hyperfinite* II_1 factor R of Murray and von Neumann, defined as the infinite tensor product $(R, \tau) = \overline{\otimes}_k (M_{2 \times 2}(\mathbb{C}), tr)_k$. By [MvN2], R is in fact the unique *approximately finite dimensional* II_1 factor, and by [C1] it is even the unique *amenable* II_1 factor. So it can be represented in many different ways, for instance as the *group measure space* II_1 factor $L^\infty(X) \rtimes \Gamma$, associated with a free ergodic measure preserving action of a countable amenable group Γ on a probability space (X, μ) . In particular, $R = L^\infty([0, 1]^\mathbb{Z}) \rtimes \mathbb{Z}$, where $\mathbb{Z} \curvearrowright X = [0, 1]^\mathbb{Z}$ is the Bernoulli action. When viewed this way, R has $D = L^\infty(X)$ as a natural *Cartan subalgebra*, i.e. a maximal abelian $*$ -subalgebra (MASA) $D \subset R$ whose normalizer generates R . By [CFW], the Cartan subalgebra of R is in fact unique, up to conjugacy by an automorphism of R . So we may represent $D \subset R$ as the infinite tensor product $\overline{\otimes}_k (D_2)_k \subset \overline{\otimes}_k (M_{2 \times 2}(\mathbb{C}))_k$, where D_2 is the diagonal subalgebra in $M_{2 \times 2}(\mathbb{C})$.

But the hyperfinite II_1 factor R also has MASAs $A \subset R$ whose normalizer is trivial, i.e. the only unitary elements $u \in R$ normalizing A , $uAu^* = A$, are the unitaries in A . Such MASAs are called *singular* and their existence was discovered in [D1]. A typical example of singular MASA is given by the subalgebra $L(\mathbb{Z}) \subset R$, generated by the canonical unitary implementing the Bernoulli action $\mathbb{Z} \curvearrowright [0, 1]^\mathbb{Z}$, in the above representation of the hyperfinite factor $R = L^\infty([0, 1]^\mathbb{Z}) \rtimes \mathbb{Z}$.

There is an *ultraproduct* procedure of constructing II_1 factors from a free ultrafilter ω on \mathbb{N} and a sequence of factors (M_m, τ) , with M_m either II_1 , or finite dimensional with $\dim M_m \nearrow \infty$ (see [W], [F]). The initial motivation for our work has been the observation that the Kadison-Singer property for $\ell^\infty \mathbb{N} \subset \mathcal{B}(\ell^2 \mathbb{N})$, as well as its paving version, are equivalent to the analogue statements for the ultrapower inclusions $D^\omega \subset R^\omega$, respectively $\Pi_\omega D_m \subset \Pi_\omega M_{m \times m}(\mathbb{C})$. Paving here means that if $x \in R^\omega$ (resp. $x \in \Pi_\omega M_{m \times m}(\mathbb{C})$) has trace preserving expectation onto D^ω (resp. $\Pi_\omega D_m$) equal to 0, then for any $\varepsilon > 0$, there exists a partition of 1 with finitely many projections p_1, \dots, p_n in D^ω (resp. $\Pi_\omega D_m$), such that $\|\sum_{i=1}^n p_i x p_i\| \leq \varepsilon \|x\|$.

The operator norm of an element y in a II_1 factor can be calculated by the formula $\|y\| = \lim_n \tau((y^* y)^{2n})^{1/2n}$. So in order to pave x one needs to control the “higher moments” $\tau((y^* y)^n)$ for $y = \sum_i p_i x p_i$. Our idea here is to approach such calculations by using a technique developed in [P8], which consists of building the paving p_i by patching together small, “infinitesimal” pieces of projections, with “incremental” control of the moments. Ideally, one wants to build the partition p_i so that to be “free independent” with respect to the given x , because then the paving diminishes the operator norm by $\sqrt{\varepsilon}$ if the mesh of the partition is $\leq \varepsilon$, due to norm calculations in [V].

In the case of the Cartan subalgebra $D \subset R$, the independence “breaks” after the 3rd moment, more precisely we show that $D^\omega \subset R^\omega$ contains finite partitions with projections p_i in D^ω that are 3-independent to x , but if x normalizes D then $x u x^* u^* = u^* x u x^*$, for any $u \in D^\omega$, so 4-independence fails in general.

Nevertheless, our approach does provide “free paving” for any ultrapower $A^\omega \subset R^\omega$ of a singular MASA $A \subset R$, in fact for any ultraproduct of singular MASAs in II_1 factors (N.B.: by [P3] any II_1 factor contains singular MASAs). Note that this result provides the first instance where the Kadison-Singer property is established for a MASA in an infinite dimensional von Neumann factor.

0.1. Theorem (Kadison-Singer for ultrapowers of singular MASAs). *Let $A_m \subset M_m$, $m \geq 1$, be a sequence of singular MASAs in II_1 factors and denote $\mathbf{A} = \Pi_\omega A_m \subset \Pi_\omega M_m = \mathbf{M}$, their ultraproduct, over a free ultrafilter ω on \mathbb{N} . Then $\mathbf{A} \subset \mathbf{M}$ satisfies the Kadison-Singer property, i.e. any pure state on \mathbf{A} has a unique state extension to \mathbf{M} . Moreover, $\mathbf{A} \subset \mathbf{M}$ has the uniform paving property: if $x \in \mathbf{M}$ has 0-expectation on \mathbf{A} , then $\forall \varepsilon > 0$, $\exists p_1, \dots, p_n$ partition of 1 with projections in \mathbf{A} , with $n \leq C\varepsilon^{-6}$ for some universal constant C , such that $\|\sum_{i=1}^n p_i x p_i\| \leq \varepsilon \|x\|$.*

As we mentioned before, the way we prove the above result is by showing that given any $x \perp \mathbf{A}$, there exists a diffuse abelian subalgebra $B_0 \subset \mathbf{A}$ which is free independent to x , i.e. any alternating word $\Pi_{i=1}^k u_i x_i$, with letters $x_i \in \{x, x^*\}$, $u_i \in B_0 \ominus \mathbb{C}$, has trace 0. The presence of “asymptotic freeness” in a MASA

$A \subset M$ characterizes in fact singularity, and for it to be satisfied, asymptotic 4-independence is actually sufficient ($B_0 \subset A^\omega$ is n -independent to $X \subset M^\omega \ominus A^\omega$ if $\tau(\prod_{i=1}^k u_i x_i) = 0$, $\forall k \leq n$, $u_i \in B_0 \ominus \mathbb{C}$, $x_i \in X$). In turn, we will show in Section 3 and 5.4.1 that existence of asymptotic 3-independence holds in any MASA.

0.2. Theorem (Characterizations of singularity for MASAs). *Let A be a MASA in a II_1 factor M . The following are equivalent:*

- 1° A is singular in M ;
- 2° A^ω is maximal amenable in M^ω ;
- 3° Given any countable set $X \subset M^\omega \ominus A^\omega$, there exists $B_0 \subset A^\omega$ diffuse such that B_0, X are free independent.
- 4° Given any countable set $X \subset M \ominus A$, there exists $B_0 \subset A^\omega$ diffuse such that B_0, X are 4-independent.

The paper is organized as follows: In Section 1 we recall Kadison-Singer classic results from [KS], on the equivalence between the unique state extension of pure states from $\ell^\infty \mathbb{N}$ to $\mathcal{B}(\ell^2 \mathbb{N})$ and the paving property, as well as other basic facts. In Section 2 we prove the equivalence between the Kadison-Singer problem and several similar statements in II_1 factor framework. In Section 3 we show that given any MASA A in a II_1 factor M , one can pave any finite set $X \subset M \ominus A$ with respect to the L^2 -norm given by the trace. This result has already been shown in ([P1]; see also A. 1 in [P7]), but we give it here a different proof, by showing that A contains finite partitions of arbitrary small mesh that are approximately 2-independent to X . In Section 4 we prove Theorem 0.1 (as Corollary 4.3). We do this by utilizing the L^2 -paving from Section 3 and the “incremental patching method” from [P8]. In Section 5 we derive Theorem 0.2 (as Theorem 5.2.1) and obtain several related results, including existence of approximate 3-independence in arbitrary MASAs (see Theorem 5.3.1). We also formulate a conjecture strengthening Kadison-Singer (see 5.7.1), and comment on a way to modify our approach towards settling it.

While we made an effort to make this paper as self-contained as possible, for the most basic results on von Neumann algebras and II_1 factors, we refer the reader to the classic books [D2], [KR].

1. PRELIMINARIES

We recall in this section a result from [KS], showing that pure states on a maximal abelian von Neumann subalgebra \mathcal{A} of a von Neumann algebra \mathcal{M} have unique state extensions to \mathcal{M} if and only if all elements in \mathcal{M} have a certain “paving property” relative to \mathcal{A} . For the reader’s convenience, we have included a proof. It

is essentially the original one from [KS], but explained in more modern terms, and adding the reformulation of paving in terms of “relative Dixmier property”. We also introduce some necessary terminology and prove some basic related results.

1.1. Notation. Let \mathcal{M} be a von Neumann algebra and $\mathcal{A} \subset \mathcal{M}$ a maximal abelian $*$ -subalgebra (hereafter abbreviated MASA) in \mathcal{M} . If $x \in \mathcal{M}$ then we denote by $C_{\mathcal{A}}(x)$ the norm closure of the convex hull of the set $\{uxu^* \mid u \in \mathcal{U}(\mathcal{A})\}$. Also, given a finite n -tuple of unitaries $V = (v_1, \dots, v_n)$ in \mathcal{A} and $y \in \mathcal{M}$, we denote $T_V(y) = n^{-1} \sum_{i=1}^n v_i y v_i^* \in C_{\mathcal{A}}(y)$. Note right away that the commutativity of \mathcal{A} implies $T_U(T_V(y)) = T_V(T_U(y))$ for any two such tuples U, V . Also, $\|T_U(y)\| \leq \|y\|$ and $T_U(a_1 y a_2) = a_1 T_U(y) a_2$, $\forall a_1, a_2 \in \mathcal{A}$ (i.e., the maps T_U are \mathcal{A} -bimodular).

1.2. Theorem (Kadison-Singer [KS]). *If $\mathcal{A} \subset \mathcal{M}$ is a MASA in a von Neumann algebra \mathcal{M} , then the following conditions are equivalent:*

- (1.2.1) *Any pure state on \mathcal{A} has a unique pure state extension to \mathcal{M} .*
- (1.2.2) $C_{\mathcal{A}}(x) \cap \mathcal{A} \neq \emptyset$, $\forall x \in \mathcal{M}$.
- (1.2.3) $C_{\mathcal{A}}(x) \cap \mathcal{A}$ is a single point set $\{E_{\mathcal{A}}(x)\}$, $\forall x \in \mathcal{M}$.
- (1.2.4) *For all $x \in \mathcal{M}$ and all $\varepsilon > 0$ there exists a finite partition of 1 with projections $q_k \in \mathcal{A}$ such that $d(\sum_k q_k x q_k, \mathcal{A}) \leq \varepsilon$.*
- (1.2.5) *For all $x \in \mathcal{M}$, there exists a unique element $E(x) \in \mathcal{A}$ with the property that $\forall \varepsilon > 0$, $\exists q_k \in \mathcal{P}(\mathcal{A})$ a finite partition of 1 such that $\|\sum_k q_k x q_k - E(x)\| \leq \varepsilon$.*

Moreover, if these conditions are satisfied then $E(x) = E_{\mathcal{A}}(x)$ and the map $E_{\mathcal{A}}$ satisfies the following additional properties:

- (i) $\overline{C_{\mathcal{A}}(x)}^w \cap \mathcal{A} = \{E_{\mathcal{A}}(x)\}$, $\forall x \in \mathcal{M}$.
- (ii) $E_{\mathcal{A}}$ is the unique conditional expectation of \mathcal{M} onto \mathcal{A} .
- (iii) Given any pure state ψ on \mathcal{A} , $\psi \circ E_{\mathcal{A}}$ is the unique state extension of ψ to \mathcal{M} , and it is a pure state.

Proof. The implication (1.2.3) \implies (1.2.2) is trivial. If (1.2.2) is satisfied and $a, b \in C_{\mathcal{A}}(x)$ are distinct, then there exist tuples U, V such that $\|T_U(x) - a\| \leq \|a - b\|/4$ and $\|T_V(x) - b\| \leq \|a - b\|/4$. But $\|T_V(T_U(x)) - a\| = \|T_V(T_U(x) - a)\| \leq \|T_U(x) - a\|$ and $\|T_U(T_V(x)) - b\| = \|T_U(T_V(x) - b)\| \leq \|T_V(x) - b\|$. Thus we have

$$\begin{aligned} \|a - b\| &\leq \|T_V(T_U(x)) - a\| + \|T_V(T_U(x)) - T_U(T_V(x))\| + \|T_U(T_V(x)) - b\| \\ &= \|T_V(T_U(x)) - a\| + \|T_U(T_V(x)) - b\| \leq \|a - b\|/4 + \|a - b\|/4 = \|a - b\|/2, \end{aligned}$$

a contradiction. This proves (1.2.2) \implies (1.2.3).

A similar argument shows that (1.2.4) and (1.2.5) are equivalent.

Assuming now (1.2.3), we prove (1.2.5) as well as the properties (i) – (iii). Let $x \in \mathcal{M}$, $\|x\| \leq 1$, and $\varepsilon > 0$. Let $u_1, \dots, u_n \in \mathcal{A}$ be so that $\|T_U(x) - E_{\mathcal{A}}(x)\| \leq \varepsilon/2$, where $U = (u_1, \dots, u_n)$. For each $i = 1, \dots, n$ let $\{e_{ij}\}_j \in \mathcal{A}$ be spectral projections of u_i such that if we denote $v_i = \sum_j \lambda_{ij} e_{ij}$, then $\|u_i - v_i\| \leq \varepsilon/4$. Thus, if we denote $V = (v_1, \dots, v_n)$, then $\|T_U(x) - T_V(x)\| \leq \varepsilon/2$ and hence $\|T_V(x) - E_{\mathcal{A}}(x)\| \leq \varepsilon$. By taking into account that if $\{q_k\}_k$ denotes a relabeling of the set of projections $\{e_{ij}\}_{i,j}$, then $\sum_k q_k T_v(x) q_k = \sum_k q_k x q_k$, we thus get

$$\|\sum_k q_k x q_k - E_{\mathcal{A}}(x)\| = \|\sum_k q_k (T_v(x) - E_{\mathcal{A}}(x)) q_k\| \leq \|T_v(x) - E_{\mathcal{A}}(x)\| \leq \varepsilon,$$

proving the existence part of (1.2.5). Since any element of the form $\sum_j p_j x p_j$, with p_1, \dots, p_n a partition with projections in \mathcal{A} , is of the form $T_W(x) \in C_{\mathcal{A}}(x)$, where $W = (w^{j-1})_{j=1}^n$, $w = \sum_{i=1}^n \lambda^{(i-1)} p_i$, where $\lambda = \exp(2\pi i/n)$, by the uniqueness in (1.2.3) we get the uniqueness part in (1.2.5) and that $E(x) = E_{\mathcal{A}}(x)$. We have also implicitly shown that (1.2.5) \implies (1.2.2).

Since \mathcal{A} is abelian (thus amenable), there exists a conditional expectation $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{A}$ (obtained by taking a Banach limit of appropriate averages T_U). By (1.2.3), for any fixed $x \in \mathcal{M}$ and any $\varepsilon > 0$, there exists a tuple $V = (v_1, \dots, v_n)$ in \mathcal{A} such that fixed $\|T_V(x) - E_{\mathcal{A}}(x)\| \leq \varepsilon$. Thus

$$\begin{aligned} \|\mathcal{E}(x) - E_{\mathcal{A}}(x)\| &= \|T_V(\mathcal{E}(x)) - E_{\mathcal{A}}(x)\| \\ &= \|\mathcal{E}(T_V(x) - E_{\mathcal{A}}(x))\| \leq \|T_V(x) - E_{\mathcal{A}}(x)\| \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\mathcal{E}(x) = E_{\mathcal{A}}(x)$, $\forall x \in \mathcal{M}$, proving (ii).

Fix now $x_0 \in \mathcal{M}$ and let $y_0 \in \overline{C_{\mathcal{A}}(x)}^w \cap \mathcal{A}$ and $\{U_{\iota}\}_{\iota \in I}$ be a net of tuples of unitaries in \mathcal{A} such that the weak limit of $\{T_{U_{\iota}}(x_0)\}_{\iota}$ is equal to y_0 . Let Lim_{ι} be a Banach limit over ι and for each $x \in \mathcal{M}$ denote $\Phi(x) = \text{Lim}_{\iota} T_{U_{\iota}}(x)$. Then $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ is linear, positive, \mathcal{A} -bimodular, $\Phi(a) = a$, $\forall a \in \mathcal{A}$, and $\Phi(x_0) = y_0$. But then $\mathcal{E}(x) = E_{\mathcal{A}}(\Phi(x))$ is a conditional expectation of \mathcal{M} onto \mathcal{A} satisfying $\mathcal{E}(x_0) = y_0$. By (ii), this forces $y_0 = E_{\mathcal{A}}(x_0)$, proving (i).

Let now ψ be a pure state on \mathcal{A} . By Gelfand-Naimark, ψ is given by the evaluation at some point in the spectrum Ω of \mathcal{A} (thus Ω is a hyperstonian compact space and $\mathcal{A} = C(\Omega)$). In particular, ψ is multiplicative and takes only the values 0, 1 on the set of projections $\mathcal{P}(\mathcal{A})$, with $\psi(1) = 1$. This implies that any state extension φ of ψ to \mathcal{M} has \mathcal{A} in its centralizer \mathcal{M}_{φ} . Indeed, because if $\psi(p) = 0$ for some $p \in \mathcal{A}$, then by the Cauchy-Schwartz inequality for φ we have $|\varphi(px)| \leq \varphi(p)^{1/2} \varphi(x^*x)^{1/2} = 0$, $|\varphi(xp)| \leq \varphi(xx^*)^{1/2} \varphi(p)^{1/2} = 0$, $\forall x \in \mathcal{M}$. Since for any projection $p \in \mathcal{A}$ we either have $\psi(p) = 0$ or $\psi(p) = 1$ and $1 \in \mathcal{M}_{\varphi}$, this shows that $\mathcal{P}(\mathcal{A}) \subset \mathcal{M}_{\varphi}$, thus all \mathcal{A} is contained in \mathcal{M}_{φ} . Hence, φ is constant on

$C_{\mathcal{A}}(x)$, which contains $E_{\mathcal{A}}(x)$ by (1.2.3), implying that $\psi(x) = \psi(E_{\mathcal{A}}(x))$. This proves (1.2.3) \implies (1.2.1) and (1.2.3) \implies (iii).

We have shown so far that (1.2.2) – (1.2.5) are equivalent and that they imply (1.2.1) and (i) – (iii). To prove the remaining implication (1.2.1) \implies (1.2.2), let $b \in \mathcal{M}_h$ and fix a point $t \in \Omega$ in the spectrum of \mathcal{A} . Letting $c_0 = \inf\{a(t) \mid a \in \mathcal{A}_h, a \geq b\}$, $c_1 = \sup\{a(t) \mid a \in \mathcal{A}_h, a \leq b\}$, we first show that condition (1.2.1) implies $c_0 = c_1$. For if not, then the maps $\psi_i : \mathcal{A} + \mathbb{C}b \rightarrow \mathbb{C}$ defined by $\psi_i(y + \alpha b) = y(t) + \alpha c_i$, $i = 0, 1$, $y \in \mathcal{A}$, $\alpha \in \mathbb{C}$, are well defined, linear and positive; thus $\|\psi_i\| = 1$ and by Hahn-Banach each ψ_i can be extended to a norm-1 linear functional $\varphi_i : \mathcal{M} \rightarrow \mathbb{C}$; we have thus obtained two states φ_0, φ_1 on \mathcal{M} , which extend the pure state t and are distinct (because $\varphi_0(b) \neq \varphi_1(b)$), contradicting (1.2.1). Let now $\varepsilon > 0$ and for each $t \in \Omega$ denote $c_t = \inf\{a(t) \mid a \in \mathcal{A}_h, a \geq b\} = \sup\{a(t) \mid a \in \mathcal{A}_h, a \leq b\}$. Let $a_t^\pm \in \mathcal{A}_h$ be such that $a_t^+ \geq b \geq a_t^-$ and $c_t + \varepsilon/2 > a_t^+(t)$, $a_t^-(t) > c_t - \varepsilon/2$. By the continuity of $a_t^\pm \in \mathcal{A} = C(\Omega)$ as a function on Ω , there exists an open-closed neighborhood Ω_t of t in Ω such that $c_t + \varepsilon/2 > a_t^+(t')$, $a_t^-(t') > c_t - \varepsilon/2$, $\forall t' \in \Omega_t$. Thus, if we denote by $p_t \in C(\Omega)$ the characteristic function of Ω_t , then p_t is a projection in \mathcal{A} satisfying

$$(c_t + \varepsilon/2)p_t \geq a_t^+ p_t \geq p_t b p_t \geq a_t^- p_t \geq (c_t - \varepsilon/2)p_t.$$

In particular, $\|p_t b p_t - c_t p_t\| \leq \varepsilon$. Since Ω is compact, there exist $t_1, \dots, t_n \in \Omega$ such that $\cup_i \Omega_{t_i} = \Omega$. If we now take q_1 to be the characteristic function of Ω_{t_1} and for each $j \geq 2$, p_j to be the characteristic function of $\Omega_j \setminus \cup_{i=1}^{j-1} \Omega_i$, viewed as a projection in \mathcal{A} , it follows that $\|\sum_j q_j b q_j - \sum_j c_{t_j} q_j\| \leq \varepsilon$ with $\sum_j c_{t_j} q_j \in C_{\mathcal{A}}(b)$. \square

1.3. Remark. The above proof actually shows that properties (1.2.1) – (1.2.4) are equivalent for any given element $x \in \mathcal{M}$. More precisely, we have proved the following “local” statement: Let \mathcal{A} be a MASA in a von Neumann algebra \mathcal{M} and let $x \in \mathcal{M}$. The following properties are equivalent:

(1.3.1) Any two state extensions on \mathcal{M} of a pure state on \mathcal{A} coincide at x .

(1.3.2) $C_{\mathcal{A}}(x) \cap \mathcal{A} \neq \emptyset$.

(1.3.3) $C_{\mathcal{A}}(x) \cap \mathcal{A}$ is a single point set $\{E_{\mathcal{A}}(x)\}$.

(1.3.4) For all $\varepsilon > 0$, there exists a finite partition of 1 with projections $q_k \in \mathcal{A}$ such that $d(\sum_k q_k x q_k, \mathcal{A}) \leq \varepsilon$.

(1.3.5) There exists a unique element $E(x) \in \mathcal{A}$ such that for all $\varepsilon > 0$, there exists a finite partition of 1 with projections $q_k \in \mathcal{A}$ such that $\|\sum_k q_k x q_k - E(x)\| \leq \varepsilon$.

Moreover, if these conditions are satisfied for x , then $E(x) = E_{\mathcal{A}}(x)$ and the following additional properties hold true:

(i) $\overline{C_{\mathcal{A}}(x)}^w \cap \mathcal{A} = \{E_{\mathcal{A}}(x)\}$.

(ii) Any conditional expectation \mathcal{E} of \mathcal{M} onto \mathcal{A} (which always exist because \mathcal{A} is abelian) satisfies $\mathcal{E}(x) = E_{\mathcal{A}}(x)$.

(iii) Any extension of a pure state ψ on \mathcal{A} to a state φ on \mathcal{M} , satisfies $\varphi(x) = \psi(E_{\mathcal{A}}(x))$.

1.4. Definitions. Let \mathcal{M} be a von Neumann algebra and $\mathcal{A} \subset \mathcal{M}$ a MASA in \mathcal{M} . We will use the following terminology:

(1.4.1) $\mathcal{A} \subset \mathcal{M}$ satisfies the *Kadison-Singer* (abbreviated *KS*) *property* if (1.2.1) is satisfied. Condition (1.2.4) is referred to as the *paving property* for $\mathcal{A} \subset \mathcal{M}$ (the term was coined in [A2]). Also, condition (1.2.3) is called the *relative Dixmier property* for $\mathcal{A} \subset \mathcal{M}$, because of its relation to a phenomenon first emphasized in [D2] (the “Dixmier averaging by unitaries”). Note that by Theorem 1.2 these three properties for $\mathcal{A} \subset \mathcal{M}$ are actually equivalent, and they imply 1.2(i) – (iii) as well.

(1.4.2) An element $x \in \mathcal{M}$ *can be paved* (over \mathcal{A}) if condition (1.3.5) is satisfied. A set $X \subset \mathcal{M}$ can be paved if each $x \in X$ can be paved. If $x \in \mathcal{M}$ can be paved and $\varepsilon > 0$, then we denote by $n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon)$ (or simply $n(x, \varepsilon)$ if no confusion is possible) the smallest number n for which there exists a partition of 1 with n projections $p_1, \dots, p_n \in \mathcal{A}$, such that $\|\sum_{i=1}^n p_i x p_i - E_{\mathcal{A}}(x)\| \leq \varepsilon \|x - E_{\mathcal{A}}(x)\|$, where $E_{\mathcal{A}}(x)$ is given by Remark 1.3.

More generally, if $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{A}$ is a conditional expectation, $x \in \mathcal{M}$ and $\varepsilon > 0$, then we say that x can be ε -*paved with respect to* \mathcal{E} , if there exists a finite partition with projections $p_1, \dots, p_n \in \mathcal{A}$ such that $\|\sum_i p_i x p_i - \mathcal{E}(x)\| \leq \varepsilon \|x - \mathcal{E}(x)\|$ and we denote by $n(\mathcal{E}; x, \varepsilon)$ the smallest number of such projections. If there exists no such finite partition, then we let $n(\mathcal{E}; x, \varepsilon) = \infty$. One should note that if $\mathcal{E}' : \mathcal{M} \rightarrow \mathcal{A}$ is another expectation, then $\|\mathcal{E}'(x) - \mathcal{E}(x)\| \leq \varepsilon \|x - \mathcal{E}(x)\|$ and $\|\sum_i p_i x p_i - \mathcal{E}'(x)\| \leq 2\varepsilon \|x - \mathcal{E}'(x)\|$, so we have $n(\mathcal{E}; x, \varepsilon) \geq n(\mathcal{E}'; x, 2\varepsilon)$. Thus, taking one expectation or another doesn't really change the nature of the function $n(\cdot; x, \varepsilon)$ and they are all “comparable” to $n(d; x, \varepsilon)$, which is by definition the smallest n for which there exists a partition of 1 with projections $p_1, \dots, p_n \in \mathcal{A}$ such that $d(\sum_i p_i x p_i, \mathcal{A}) \leq \varepsilon d(x, \mathcal{A})$. In case it is clear from the context what expectations we take, then \mathcal{E} will not be mentioned, and we just use the notation $n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon)$. Also, in case there exists a normal conditional expectation of \mathcal{M} onto \mathcal{A} (e.g. when $\mathcal{M} = M$ is a finite von Neumann algebra), then ε -pavings are always considered with respect to this expectation.

(1.4.3) A set $X \subset \mathcal{M}$ has the *uniform paving property* (over \mathcal{A}) if it can be paved and if $n(\mathcal{A} \subset \mathcal{M}; X, \varepsilon) \stackrel{\text{def}}{=} \sup\{n(x, \varepsilon) \mid x \in X\}$ is finite, $\forall \varepsilon > 0$. If this holds true for $X = \mathcal{M}$, we say that $\mathcal{A} \subset \mathcal{M}$ has the uniform paving property and use the

notation $n(\mathcal{A} \subset \mathcal{M}; \varepsilon)$ for $n(\mathcal{A} \subset \mathcal{M}; \mathcal{M}, \varepsilon)$. We call this function the *paving size* of $\mathcal{A} \subset \mathcal{M}$. We will be interested in the order of magnitude of the (decreasing) functions $n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon)$, i.e. up to the equivalence relation $f(\varepsilon) \sim g(\varepsilon)$ for functions f, g requiring the existence of positive constants $0 < c < C < \infty$ such that $c \leq f(\varepsilon)/g(\varepsilon) \leq C$, $\forall \varepsilon > 0$. As we will see below, the uniform paving property appears naturally in this context, being often equivalent to the usual paving property (notably in the case $\mathcal{D} \subset \mathcal{B}$), a fact first pointed out by Anderson in [A1], [A2].

(1.4.4) Let $\mathcal{D} = \ell^\infty \mathbb{N}$ be the diagonal MASA in the algebra $\mathcal{B} = \mathcal{B}(\ell^2 \mathbb{N})$ of all linear bounded operators on the Hilbert space $\ell^2 \mathbb{N}$. It is easy to see that the conditional expectation $\mathcal{B} \ni (\alpha_{jk})_{j,k \in \mathbb{N}} \mapsto (\alpha_{kk})_k \in \mathcal{D}$ is the unique conditional expectation of \mathcal{B} onto \mathcal{D} and that it is normal. We use the terminology “*the classic Kadison-Singer problem*” for the question of whether $\mathcal{D} \subset \mathcal{B}$ has the KS property. By Theorem 1.2, this property is equivalent to the paving property for $\mathcal{D} \subset \mathcal{B}$. The terminology “*Kadison-Singer conjecture*” is sometimes used for the statement predicting that the KS property does hold true for this inclusion, despite the fact that, in their paper, Kadison and Singer expressed the belief that the property doesn’t actually hold true for $\mathcal{D} \subset \mathcal{B}$

The next result summarizes some well known paving properties, notably J. Anderson’s observations that uniform paving for $\mathcal{D} \subset \mathcal{B}$ is equivalent to paving and that the classic Kadison-Singer problem is equivalent to $D_k \subset M_{k \times k}(\mathbb{C})$ having uniformly bounded paving size (see [A1], [A2]).

1.5. Proposition. *0° Let \mathcal{A} be a MASA in the von Neumann algebra \mathcal{M} , $p \in \mathcal{P}(\mathcal{A})$ a projection and $\mathcal{A} \subset \mathcal{N} \subset \mathcal{M}$ an intermediate von Neumann algebra. If $x \in p\mathcal{M}p$ (respectively $x \in \mathcal{N}$) then $n(\mathcal{A}p \subset p\mathcal{M}p; x, \varepsilon) = n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon)$ (resp. $n(\mathcal{A} \subset \mathcal{N}; x, \varepsilon) = n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon)$).*

1° Let $\{\mathcal{A}_i \subset \mathcal{M}_i\}_i$ be a family of MASAs in von Neumann algebras and for each i let $x_i \in \mathcal{M}_i$, $\|x_i\| \leq 1$. Denote $\mathcal{A} = \oplus_i \mathcal{A}_i$, $\mathcal{M} = \oplus_i \mathcal{M}_i$, $x = \oplus_i x_i \in \mathcal{M}$. Then $n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon) = \sup_i n(\mathcal{A}_i \subset \mathcal{M}_i; x_i, \varepsilon)$ and $n(\mathcal{A} \subset \mathcal{M}; \varepsilon) = \sup_i n(\mathcal{A}_i \subset \mathcal{M}_i; \varepsilon)$, $\forall \varepsilon > 0$.

2° If a MASA \mathcal{A} in a von Neumann algebra \mathcal{M} has the property that there exists a sequence of mutually orthogonal projections $p_n \in \mathcal{A}$ with embeddings $\theta_n : \mathcal{M} \hookrightarrow p_n \mathcal{M} p_n$ such that $\theta_n(\mathcal{A}) = \mathcal{A} p_n$, $\forall n$, then $\mathcal{A} \subset \mathcal{M}$ has the paving property iff it has the uniform paving property.

3° The diagonal MASA $\mathcal{D} = \ell^\infty \mathbb{N}$ in the algebra of all linear bounded operators $\mathcal{B} = \mathcal{B}(\ell^2 \mathbb{N})$ on the Hilbert space $\ell^2 \mathbb{N}$, has the paving property iff it has the uniform paving property. Moreover, $n(\mathcal{D} \subset \mathcal{B}; \varepsilon) = \sup_k n(D_k \subset M_{k \times k}(\mathbb{C}); \varepsilon)$, $\forall \varepsilon > 0$.

4° If \mathcal{A} is a MASA in a von Neumann algebra \mathcal{M} , with $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{A}$ an expectation, and $1 > \varepsilon > 0$, then we have $\sup\{n(\mathcal{E}; x, \varepsilon^2) \mid x \in \mathcal{M}\} \leq (\sup\{n(\mathcal{E}; y, \varepsilon) \mid y \in \mathcal{M}\})^2$. Thus, in order for $\mathcal{A} \subset \mathcal{M}$ to have the uniform paving property, it is sufficient that for some $\varepsilon < 1$ we have $\sup\{n(\mathcal{E}; y, \varepsilon) \mid y \in \mathcal{M}\} < \infty$.

Proof. Parts 0° and 1° are trivial and 2° is an immediate consequence of 1°. Then 2° implies the equivalence in the first part of 3°.

To establish the formula in 3°, note that if a sequence of projections q_n in $\ell^\infty \mathbb{N}$ is convergent in the weak operator topology to some element $q \in \ell^\infty \mathbb{N}$, then q is itself a projection and q_n converges to q in the strong operator topology as well.

The inequality $n(\mathcal{D} \subset \mathcal{B}; \varepsilon) \geq \sup_k n(D_k \subset M_{k \times k}(\mathbb{C}); \varepsilon)$ is trivial because the right hand side is equal to $n(\oplus_k D_k \subset \oplus_k M_{k \times k}(\mathbb{C}); \varepsilon)$ and one can embed $\oplus_k M_{k \times k}(\mathbb{C})$ into \mathcal{B} in a way that takes $\oplus_k D_k$ onto \mathcal{D} .

For the inequality \leq let $T \in \mathcal{B}$ be so that $\|T\| \leq 1$ and T has 0 on the diagonal. Let $P_k \in \mathcal{D} = \ell^\infty \mathbb{N}$ be the projection onto the first k coordinates. Let $\{p_{k,j}\}_j$ be a partition of P_k into $n = \sup_k n(D_k \subset M_{k \times k}(\mathbb{C}); \varepsilon)$ projections such that $\|\sum_j p_{k,j} T p_{k,j}\| \leq \varepsilon$. Let $k_1 < k_2 < \dots$ be a subsequence such that $\{p_{k_m,j}\}_m$ is weakly convergent for each $j = 1, 2, \dots, n$ and denote by q_j the corresponding weak limit. By the above observation, q_j are projections and the convergence are in fact *so*. Also, since $\sum_j p_{k_m,j} = P_{k_m}$ and P_{k_m} *so*-converges to $1_{\mathcal{B}}$, it follows that $\sum_j q_j = 1$ as well. Thus, $\{q_j\}_j$ is a partition of 1 by n projections and since $\{\sum_j p_{k_m,j} T p_{k_m,j}\}_m$ is *so*-convergent to $\sum_j q_j T q_j$ and the operator norm is inferior semicontinuous with respect to the *so*-convergence, it follow that $\|\sum_j q_j T q_j\| \leq \limsup_m \|\sum_j p_{k_m,j} T p_{k_m,j}\| \leq \varepsilon$.

Finally, part 4° is immediate from the definitions. \square

1.6. Remark. While the classic Kadison-Singer problem is still open, one should point out that a large number of beautiful paving results have been obtained over the years, showing that the equivalent conditions 1.3 are satisfied for many classes of operators $x \in \mathcal{B}(\ell^2 \mathbb{Z})$. Thus, it is shown in [A2] that if x is in the C^* -algebra for the reduced C^* -algebra of the group \mathbb{Z} , $C_r^*(\mathbb{Z})$, i.e. in the operator norm-closure of the span of the range of the left regular representation λ of the group \mathbb{Z} , then x can be paved. In [BeHKW] it is shown that matrices with non-negative entries in $M_{n \times n}(\mathbb{C})$ can be paved, while in [BT] it is shown that if an element x in the weak closure $L(\mathbb{Z}) \simeq L^\infty(\mathbb{T}) \subset \mathcal{B}(\ell^2 \mathbb{Z})$ has Fourier coefficients satisfying certain growth properties, then x can be paved. Also, a number of results have been obtained in [AkA], [A1], [A2], [CaFTW], etc, showing that in order to solve the paving conjecture, it is sufficient to be able to pave certain particular classes of elements (e.g. projections with small diagonal entries in [AkA]).

2. KADISON-SINGER IN II_1 FACTOR FRAMEWORK

We prove in this section that the KS property for the inclusion of the diagonal MASA $\mathcal{D} = \ell^\infty \mathbb{N}$ into the algebra $\mathcal{B} = \mathcal{B}(\ell^2 \mathbb{N})$, of all linear bounded operators on the Hilbert space $\ell^2 \mathbb{N}$, is equivalent to the KS property of MASAs in II_1 factors obtained as ultraproducts of certain Cartan inclusions. Also, we use a dilation trick to prove that in order to pave arbitrary elements in an ultraproduct of inclusions of MASAs, it is sufficient to pave projections that expect on scalars.

From now on, we fix once for all an (arbitrary) free ultrafilter ω on \mathbb{N} . All finite von Neumann algebras that we consider are assumed equipped with a faithful normal trace state, generically denoted by τ (unless otherwise specified).

If M_n , $n \geq 1$, is a sequence of finite von Neumann algebras then, we denote by $\Pi_\omega M_n$ their ω -ultraproduct, i.e., the finite von Neumann algebra obtained as the quotient of $\oplus_n M_n$ by its ideal $\mathcal{I}_\omega = \{(x_n) \mid \lim_\omega \tau(x_n^* x_n) = 0\}$, endowed with the trace $\tau(y) = \lim_\omega \tau(y_n)$, where $(y_n)_n \in \oplus_n M_n$ is in the class $y \in \oplus_n M_n / \mathcal{I}_\omega$ ([W]). Recall that if M_n are factors and $\dim M_n \rightarrow \infty$, then $\Pi_\omega M_n$ is a II_1 factor ([W], [F]) and that if $A_n \subset M_n$ are MASAs, $n \geq 1$, then $\Pi_\omega A_n$ is a MASA in $\Pi_\omega M_n$ (see e.g. [P1]). If $A \subset M$ is a MASA in a finite von Neumann algebra, then $A^\omega \subset M^\omega$ denotes its ω -ultrapower, i.e. the ultraproduct of infinitely many copies of $A \subset M$. Note that M naturally embeds into M^ω , as the von Neumann subalgebra of constant sequences.

Recall that a *Cartan subalgebra* A in a finite von Neumann algebra M is a MASA in M whose normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M , i.e. $\mathcal{N}(A)'' = M$ (see [FM]).

2.1. Notations (a) We denote the inclusion $\Pi_\omega D_n \subset \Pi_\omega M_{n \times n}(\mathbb{C})$ by $\mathbf{D}(\omega) \subset \mathbf{M}(\omega)$, or simply $\mathbf{D} \subset \mathbf{M}$. Note that given any sequence of MASAs $A_n \subset M_{n \times n}(\mathbb{C})$, the von Neumann algebra $\Pi_n A_n \in \mathbf{M}$ is unitary conjugate to \mathbf{D} in \mathbf{M} . One should point out that \mathbf{D} is not a Cartan subalgebra in \mathbf{M} , in fact \mathbf{M} has no Cartan subalgebras (cf. [P2]; see 2.3.2° below). Also, \mathbf{D}, \mathbf{M} are non-separable (cf. [F]).

(b) We represent the hyperfinite II_1 factor R as the infinite tensor product $\overline{\otimes}_n (M_{2 \times 2}(\mathbb{C}), tr)_n$, where tr is the normalized trace on $M_{2 \times 2}(\mathbb{C})$. Also, we denote by $D \subset R$ the Cartan subalgebra obtained as the infinite tensor product of the diagonals $D_2 \subset M_{2 \times 2}(\mathbb{C})$. Recall that any other Cartan subalgebra $A \subset R$ is conjugate to D by an automorphism of R (cf [CFW]). Thus, if $D^\omega \subset R^\omega$ is the ω -ultrapower of $D \subset R$, then any ultraproduct $\Pi_\omega A_n \subset R^\omega$, with $A_n \subset R$ Cartan subalgebras, is conjugate to D^ω by an automorphism $\theta = (\theta_n)_n$ of R^ω , where $\theta_n \in \text{Aut}(R)$ is so that $\theta_n(A_n) = D$. We denote by $\mathbf{R} \subset R^\omega$ the von Neumann algebra $D^\omega \vee R$, generated by D^ω and R , or equivalently by D^ω and $\mathcal{N}_R(D)$.

If $\Gamma \subset \mathcal{N}_R(D)$ is any countable subgroup generating the hyperfinite equivalence

relation \mathcal{R} associated with $D \subset R$ (cf [FM]), then \mathbf{R} is generated by D^ω and Γ . Moreover, if Γ acts freely on D , then Γ acts freely on D^ω as well and so we can view \mathbf{R} as the crossed product $D^\omega \rtimes \Gamma$. Finally, note that \mathbf{R} is an amenable II_1 von Neumann algebra, but not a factor, in fact any sequence $(a_n)_n \in D^\omega$ with $a_n \in 1 \otimes_{j \geq k_n} (D_2)_j$ for some $k_n \rightarrow \infty$, lies in $R' \cap D^\omega = \mathbf{R}' \cap D^\omega = \mathcal{Z}(\mathbf{R})$ (the center of \mathbf{R}).

Note that, while D^ω is Cartan in \mathbf{R} , D^ω is not Cartan in R^ω , in fact R^ω has no Cartan subalgebras (by [P2]; see 2.3.2° below).

2.2. Theorem. *$\mathcal{D} \subset \mathcal{B}$ has the KS property (equivalently, the paving property) if and only if $\mathbf{D} \subset \mathbf{M}$ (resp. $D^\omega \subset R^\omega$, resp. $D^\omega \subset \mathbf{R}$) has this property. Moreover, all these inclusions have the same paving size (whether finite or infinite):*

$$(2.2.1) \quad n(\mathcal{D} \subset \mathcal{B}; \varepsilon) = n(\mathbf{D} \subset \mathbf{M}; \varepsilon) = n(D^\omega \subset R^\omega; \varepsilon) = n(D^\omega \subset \mathbf{R}; \varepsilon).$$

They also have the same paving size as the Cartan subalgebra inclusions $\mathbf{D} \subset \mathcal{N}_{\mathbf{M}}(\mathbf{D})''$ and $D^\omega \subset \mathcal{N}_{R^\omega}(D^\omega)''$.

Proof. Consider the inclusion $A_0 = \bigoplus_{n=1}^\infty D_n \subset \bigoplus_{n=1}^\infty M_{n \times n}(\mathbb{C}) = M_0$ and note that by 1.5.1° and 1.5.3° we have

$$n(\mathcal{D} \subset \mathcal{B}; \varepsilon) = \sup_n n(D_n \subset M_{n \times n}(\mathbb{C}); \varepsilon) = n(A_0 \subset M_0; \varepsilon).$$

Embed now A_0 into D and then extend this to an embedding of M_0 into R so that the matrix units of each direct summand $M_{n \times n}(\mathbb{C})$ are in the normalizing groupoid of $D \subset R$ (this is possible because D is Cartan in R ; in fact, semiregularity is sufficient). Note that this implies M_0 and D make a commuting square, i.e. $E_{M_0} E_D = E_D E_{M_0} = E_{A_0}$. Also, we trivially have

$$n(D^\omega \subset R^\omega; \varepsilon) \geq n(D^\omega \subset \mathbf{R}; \varepsilon) \geq n(D^\omega \subset \mathbf{R}; R, \varepsilon) \geq n(D^\omega \subset \mathbf{R}; M_0, \varepsilon).$$

Let $x = (x_n)_n \in M_0 \subset R$ be so that $E_{D^\omega}(x) = E_{A_0}(x) = 0$ and note that $n(D^\omega \subset \mathbf{R}; x, \varepsilon) = \sup_n n(D^\omega \subset \mathbf{R}; x_n, \varepsilon)$. Let $s_n \in D \subset D^\omega$ denote the support projection of $M_{n \times n}(\mathbb{C})$ in $A_0 \subset D$. Each $D^\omega s_n \subset (M_0 \vee D^\omega) s_n$ is of the form $C(\Omega) \subset C(\Omega) \otimes M_{n \times n}(\mathbb{C})$. Thus, if $p_1, \dots, p_m \in \mathcal{P}(D^\omega)$ is a partition of 1 such that $\|\sum_i p_i x p_i\| \leq \varepsilon$, then the evaluation at any point $t \in \Omega$ of $p_i s_n$, $1 \leq i \leq m$, gives a partition of $1_{M_{n \times n}(\mathbb{C})}$ with m projections q_i in D_n such that $\|\sum_{i=1}^m q_i x_n q_i\| \leq \varepsilon$. This shows that $n(D^\omega \subset \mathbf{R}; M_0, \varepsilon) \geq n(A_0 \subset M_0; \varepsilon)$. Since the latter is equal to $n(\mathcal{D} \subset \mathcal{B}; \varepsilon)$ and to $\sup_n n(D_n \subset M_{n \times n}(\mathbb{C}); \varepsilon)$, in order to end the proof of the fact

that $\mathcal{D} \subset \mathcal{B}$, $D^\omega \subset R^\omega$, $D^\omega \subset \mathbf{R}$ (as well as $A_0 \subset M_0$) have the same paving size, it is sufficient to show that $\sup_n n(D_n \subset M_{n \times n}(\mathbb{C}); \varepsilon) \geq n(D^\omega \subset R^\omega; \varepsilon)$.

To this end, let $x = (x_n)_n \in R^\omega \ominus D^\omega$ and note that one can take each x_n to belong to $R \ominus D$ and such that $\|x_n\| \leq \|x\| + c_n$, $\forall n$, for some $c_n \rightarrow 0$. Moreover, since there exists an increasing sequence of $2^k \times 2^k$ matrix subalgebras $M_{2^k} \subset R$ with diagonal subalgebra $D_{2^k} \subset D$ making a commuting square with $D \subset R$ such that $\overline{\cup_k M_{2^k}}^w = R$ and $\overline{\cup_k D_{2^k}}^w = D$, we may replace each x_n by $E_{M_{2^{k_n}}}(x_n)$, and thus assume $x_n \in M_{2^{k_n}} \ominus D_{2^{k_n}}$, $\forall n$. Let $\{q_j^n\}_j \subset D_{2^{k_n}}$ be a partition of 1 with $K = \sup_n n(D_n \subset M_{n \times n}(\mathbb{C}); \varepsilon)$ projections such that $\|\sum_{j=1}^K q_j^n x_n q_j^n\| \leq \varepsilon \|x_n\|$. If we denote by $q_j = (q_j^n)_n \in D^\omega$, it follows that $\|\sum_{j=1}^K q_j x q_j\| \leq \varepsilon \|x\|$. This proves the desired inequality.

Let now $\varepsilon > 0$ and assume $m = n(\mathcal{D} \subset \mathcal{B}; \varepsilon) = \sup_n n(D_n \subset M_{n \times n}(\mathbb{C}); \varepsilon)$ is finite. Any $x \in \mathbf{M}$ with $\|x\| \leq 1$ and $E_{\mathbf{D}}(x) = 0$ can be represented by a sequence $x = (x_n)_n$ with $x_n \in M_{n \times n}(\mathbb{C})$ such that $\|x_n\| \leq 1 + c_n$, $E_{D_n}(x_n) = 0$, for some $c_n \rightarrow 0$. For each n there exists a partition of 1 with projections p_j^n , $1 \leq j \leq m$, such that $\|\sum_{j=1}^m p_j^n x_n p_j^n\| \leq \varepsilon(1 + c_n)$. But then $p_j = (p_j^n)_n \in \mathbf{D}$ gives a partition of 1 satisfying $\|\sum_{j=1}^m p_j x p_j\| \leq \varepsilon$. Thus, $\sup_n n(D_n \subset M_{n \times n}(\mathbb{C}); \varepsilon) \geq n(\mathbf{D} \subset \mathbf{M}; \varepsilon)$.

Conversely, assume $m = n(\mathbf{D} \subset \mathbf{M}; \varepsilon)$ is finite. Let x be an element in $M_{k \times k}(\mathbb{C})$, for some $k \geq 1$, with $\|x\| \leq 1$, $E_{D_k}(x) = 0$. For each n larger than k , embed $M_{k \times k}(\mathbb{C})$ into $M_{n \times n}(\mathbb{C})$ by first letting $n = kd_n + r_n$, with $d_n, r_n \in \mathbb{N}$, $r_n < k$, then letting $s_j^n \in D_n$ be mutually orthogonal projections of trace k/n , and then identifying $D_k \subset M_{k \times k}(\mathbb{C})$ with $D_n s_j^n \subset s_j^n M_{n \times n}(\mathbb{C}) s_j^n$ via some isomorphism θ_j^n , for each $j = 1, \dots, d_n$, and mapping diagonally

$$M_{k \times k}(\mathbb{C}) \ni y \mapsto \theta^n(y) \stackrel{\text{def}}{=} \sum_j \theta_j^n(y) \in \sum_{j=1}^{d_n} s_j^n M_{n \times n}(\mathbb{C}) s_j^n \subset M_{n \times n}(\mathbb{C}).$$

Then consider the embedding $\theta : M_{k \times k}(\mathbb{C}) \rightarrow \mathbf{M}$, by $\theta(y) = (\theta^n(y))_n$. Let $p_1, \dots, p_m \in \mathcal{P}(\mathbf{D})$ be a partition of 1 such that $\|\sum_i p_i \theta(x) p_i\| \leq \varepsilon$. One can then choose representing sequences $p_i = (p_i^n)_n$, with $p_i^n \in \mathcal{P}(D_n)$, $1 \leq i \leq m$, a partition of 1 for each n . We claim that for any $\delta > 0$ there exist n and $j \in \{1, \dots, d_n\}$, such that $\|\sum_i p_i^n s_j^n \theta_j^n(x) p_j^n s_j^n\| < \varepsilon + \delta$.

Indeed, for if not then for every n and $j = 1, \dots, d_n$ the spectral projection of $|\sum_i p_i^n s_j^n \theta_j^n(x) p_j^n s_j^n|$ corresponding to the interval $[\varepsilon + \delta, 1]$ is non-zero, thus having trace at least $1/n$. Since $|\sum_i p_i^n \theta^n(x) p_i^n| = \sum_{j=1}^{d_n} |\sum_i p_i^n s_j^n \theta_j^n(x) p_j^n s_j^n|$, the spectral projection corresponding to $[\varepsilon + \delta, 1]$ of $|\sum_i p_i^n \theta^n(x) p_i^n|$ has trace $\geq d_n/n$. But this implies that the spectral projection corresponding to $[\varepsilon + \delta/2, 1]$ of $|\sum_i p_i \theta(x) p_i| = (|\sum_i p_i^n \theta^n(x) p_i^n|)_n \in \mathbf{M}$ has trace $\geq \lim_n d_n/n = 1/k$. Thus $\|\sum_i p_i \theta(x) p_i\| \geq \varepsilon + \delta/2$, a contradiction.

If we now choose some n and $j \in \{1, \dots, d_n\}$ satisfying $\|\Sigma_i p_i^n s_j^n \theta_j^n(x) p_j^n s_j^n\| < \varepsilon + \delta$, and let $q_i, 1 \leq i \leq m$, be the pre-image in D_k of the partition $\{p_i^n s_j^n\}_i$ via θ^n , then $\|\Sigma_{i=1}^m q_i x q_i\| < \varepsilon + \delta$. Letting $\delta = 1/n, n = 1, 2, \dots$, we obtain a sequence of partitions of 1 by projections $\{q_i(n), \dots, q_m(n)\}_n$ in D_k satisfying $\|\Sigma_{i=1}^m q_i x q_i\| < \varepsilon + 1/n$. But the unit ball of D_k is compact in the operator norm, so by taking the limit over some subsequence, we get a partition of 1 with projections $q_1, \dots, q_m \in D_k$ with $\|\Sigma_i q_i x q_i\| \leq \varepsilon$. Thus, $m \geq n(D_k \subset M_k; \varepsilon)$, and since k was arbitrary, $m \geq \sup_k n(D_k \subset M_k; \varepsilon)$.

Finally, since $D^\omega \subset \mathbf{R} \subset \mathcal{N}(D^\omega)'' \subset R^\omega$, we have

$$n(D^\omega \subset \mathbf{R}; \varepsilon) \leq n(D^\omega \subset \mathcal{N}_{\mathbf{M}}(\mathbf{D})''; \varepsilon) \leq n(D^\omega \subset R^\omega; \varepsilon),$$

and since the first and last terms are equal, they must all be equal. Similarly, $\mathbf{D} \subset \mathcal{N}(\mathbf{D})'' \subset \mathbf{M}$ implies $n(\mathbf{D} \subset \mathcal{N}(\mathbf{D})''; \varepsilon) \leq n(\mathbf{D} \subset \mathbf{M}; \varepsilon)$, while arguments above show that $\sup_n n(D_n \subset M_{n \times n}(\mathbb{C}); \varepsilon) \leq n(\mathbf{D} \subset \mathcal{N}(\mathbf{D})''; \varepsilon)$, with the first of these terms equal to $n(\mathbf{D} \subset \mathbf{M}; \varepsilon)$. □

If M is a finite von Neumann algebra with its faithful normal trace state τ , then we denote by $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$, the L^2 (or Hilbert) norm given by the trace. We denote by $L^2 M$ the Hilbert space obtained by completing M in this L^2 norm and view M in its *standard representation*, as left multiplication representation on $L^2 M$. We also use the notation $L^1 M$ for the completion of M in the norm $\|x\|_1 = \tau(|x|)$. We view the elements in $L^2 M$ (resp. $L^1 M$) as square summable (resp. summable) operators affiliated with $M \subset \mathcal{B}(L^2 M)$, in the usual way. All self-adjoint elements affiliated with M (in particular elements in $L^2 M$, $L^1 M$) have spectral decomposition belonging to M and they can be multiplied. In particular, we have $L^2 M \cdot L^2 M = L^1 M$.

A finite von Neumann algebra M with a normal faithful trace is *separable* if it is separable with respect to the $\|\cdot\|_2$ -norm given by the trace. This condition is easily seen to be equivalent to M being countably generated. A von Neumann algebra is *diffuse* if it has no minimal (non zero) projection. Any abelian von Neumann algebra A which is diffuse and separable is isomorphic to $L^\infty([0, 1])$ (or to $L^\infty(\mathbb{T})$). Moreover, if A is endowed with a faithful normal state τ , then the isomorphism $A \simeq L^\infty([0, 1])$ can be taken so that to carry τ onto the integral $\int \cdot d\mu$, where μ is the Lebesgue measure on $[0, 1]$.

It is well known that all separable diffuse abelian von Neumann subalgebras in an ultraproduct II_1 factor are unitary conjugate (see e.g. [P2]). We will show below that any II_1 factor M that has this property will automatically have several other properties, like absence of Cartan subalgebras (already noticed in [P2]) and the

fact that in order to pave arbitrary elements over a MASA in M , it is sufficient to pave projections that expect on scalar multiples of 1. (Note that in Anderson's formulation of the KS problem as the uniform paving property in $M_{k \times k}(\mathbb{C})$, $k \nearrow \infty$, the reduction of the problem to paving special elements, such as projections with constant diagonal, has been subject of much study, see [AkA], [A2], [CaFTW], etc).

2.3. Proposition. *1°. Assume a II_1 factor M has the property that given any projection $p \in M$, any two separable diffuse abelian von Neumann subalgebras of pMp are unitary conjugate. Then M satisfies the following properties*

(a) *Given any MASA A in M , there exists a diffuse abelian von Neumann subalgebra $B_0 \subset M$ perpendicular to A .*

(b) *M has no separable MASAs and no Cartan subalgebras.*

(c) *If A is a MASA in M , then $A \subset M$ has the paving property iff any projection in M that expects on a scalar multiple of a projection in A can be paved. Moreover, if \mathcal{P}_0 denotes the set of such projections, then the paving size $n(\varepsilon)$ of $A \subset M$ satisfies $n(\varepsilon) \leq n(A \subset M; \mathcal{P}_0, \varepsilon/50)^2(\varepsilon^{-1} + 1)^2$.*

2° *If $\{M_n\}_n$ is a sequence of finite von Neumann factors with $\dim M_n \rightarrow \infty$, then the ultraproduct II_1 factor $M = \Pi_\omega M_n$ satisfies the assumption in part 1°, i.e., given any projection $p \in M$, any two separable diffuse abelian von Neumann subalgebras of pMp are unitary conjugate. Thus, $M = \Pi_\omega M_n$ satisfies properties (a), (b), (c) as well.*

Proof. 2° is well known (see e.g. Lemma 7.1 in [P2]).

1° Part (a) is shown in the proof of Theorem 7.3 in [P2], but let us recall the argument here for completeness. Let $D \subset A$ be a separable diffuse von Neumann subalgebra. Since any two separable diffuse abelian subalgebras in M are unitary conjugate and since M contains copies of the hyperfinite II_1 factor, we may assume D is the Cartan subalgebra of such a subfactor $R \subset M$, represented as $D = D_2^{\otimes \infty} \subset M_{2 \times 2}(\mathbb{C})^{\otimes \infty} = R$. Let $D_2^0 \subset M_{2 \times 2}(\mathbb{C})$ be a maximal abelian subalgebra of $M_{2 \times 2}(\mathbb{C})$ that is perpendicular to D_2 and denote $D^0 = D_2^{0 \otimes \infty} \subset R$. Then $D \perp D^0$ and since both D, D^0 are MASAs in R , we have $E_{D' \cap M}(D^0) = E_{D' \cap R}(D^0) = E_D(D^0) = \mathbb{C}$, i.e. $D^0 \perp D' \cap M \supset A$, proving (a).

To prove (b), let A be a MASA in M . If A is separable, then it has a diffuse proper subalgebra, $A_0 \subset A$, which cannot be unitary conjugate to A because it is not a MASA. Moreover, by part (a), there exist separable diffuse abelian subalgebras D, D^0 in M such that $D \subset A$ and $D^0 \perp A$. Let $u \in \mathcal{U}(M)$ be so that $uD u^* = D^0$. Then u is perpendicular to the normalizer of A in M . Indeed, because for any $v \in \mathcal{N}_M(A)$ and any partition $p_i \in D$ of mesh $\leq \varepsilon$, we have

$$|\tau(uv)|^2 = |\tau(\sum_i p_i u v p_i)|^2 \leq \|\sum_i p_i u v p_i\|_2^2 = \sum_i \tau(u^* p_i u v p_i v^*) = \sum_i \tau(p_i)^2 \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\tau(uv) = 0$. Thus $u \perp \mathcal{N}_M(A)''$.

To prove (c), note first that for any MASA in a Π_1 factor M and any $x \in M$, the paving size over A of any element $x \in M$ behaves well with respect to scalar translations and multiplications:

$$(1) \ n(x, \varepsilon) = n(x + \alpha 1, \varepsilon), \ n(\alpha x, \varepsilon) = n(x, \varepsilon/|\alpha|).$$

Now note that if $y \in M$ is a δ -perturbation of $x \in M$, then any ε -paving of y gives a $\varepsilon + \delta$ paving of x , more precisely:

$$(2) \text{ If } \|x - y\| \leq \delta(1 + \varepsilon)^{-1} \|x - E_A(x)\| \text{ and } \|\sum_i p_i (y - E_A(y)) p_i\| \leq \varepsilon \|y - E_A(y)\|, \text{ then}$$

$$\begin{aligned} & \|\sum_i p_i (x - E_A(x)) p_i\| \\ & \leq \|(x - y) - E_A(x - y)\| + \|\sum_i p_i (y - E_A(y)) p_i\| \\ & \leq \|(x - y) - E_A(x - y)\| + \varepsilon \|y - E_A(y)\| \\ & \leq (1 + \varepsilon) \|(x - y) - E_A(x - y)\| + \varepsilon \|x - E_A(x)\| \\ & \leq (\varepsilon + \delta) \|x - E_A(x)\| \end{aligned}$$

(3) If $x_1, x_2 \in M$ can be paved, then $x_1 + x_2$ can be paved. More specifically, if $x = y_1 + i y_2$, $y_i = y_i^*$, is the decomposition of x into its real and imaginary parts, then $\|y_i\| \leq \|x\|$ and so if $p_i, q_j \in \mathcal{P}(A)$ are partitions of 1 such that $\|\sum_j p_j^i y_i p_j^i - E_A(y_i)\| \leq \varepsilon \|y_i - E_A(y_i)\|$, $i = 1, 2$, then the partition $\{p_k\}_k = \{p_i^1 p_j^2\}_{i,j}$ satisfies

$$\begin{aligned} \|\sum_k p_k x p_k - E_A(x)\| & \leq \|\sum_k p_k y_1 p_k - E_A(y_1)\| + \|\sum_k p_k y_2 p_k - E_A(y_2)\| \\ & \leq \|\sum_j p_j^1 y_1 p_j^1 - E_A(y_1)\| + \|\sum_j p_j^2 y_2 p_j^2 - E_A(y_2)\| \\ & \leq \varepsilon \|y_1 - E_A(y_1)\| + \varepsilon \|y_2 - E_A(y_2)\| \leq 2\varepsilon \|x - E_A(x)\|. \end{aligned}$$

Thus, $n(x, 2\varepsilon) \leq n(\Re x, \varepsilon) n(\Im x, \varepsilon)$.

Let now $x = x^* \in M$ with $E_A(x) = 0$ and $\|x\| = 1$. Note that $y_0 = 12^{-1}(x + 5\|x\|)$ satisfies $1/3 \leq y_0 \leq 1/2$, thus $1/3 \leq E_A(y_0) \leq 1/2$. Denote by e_k be the spectral projection of $E_A(y_0)$ corresponding to the interval $[1/3 + (k-1)\varepsilon/6, 1/3 + k\varepsilon/6)$ and note that there are $\leq \varepsilon^{-1} + 1$ many such non-zero projections. If we

denote $t_k = 1/3 + (k-1)\varepsilon/6$, $a_k = E_A(y_0)e_k$, then $t_k e_k \geq a_k \geq t_{k-1} e_k$. Thus, if we denote $b = \sum_k t_{k-1}^{-1} a_k$ then $1 \leq b \leq 1 + \varepsilon/2$ and $(1 + \varepsilon/2)^{-1/2} \leq b^{-1/2} \leq 1$. Notice now that $y = b^{-1/2} y_0 b^{-1/2}$ satisfies $E_A(y) = \sum_k t_k e_k$, $1/3 - \varepsilon/12 \leq y \leq 1/2$ and $\|y_0 - y\| \leq 2\|1 - b^{-1/2}\|\|y_0\| \leq \|1 - b^{-1/2}\| \leq \varepsilon/4$.

We now split each e_k into the sum of four projections $e_k^j \in A e_k$, of equal trace $\tau(e_k^j) = \tau(e_k)/4$, $e_k = \sum_{j=1}^4 e_k^j$, and denote $y_{kj} = e_k^j y e_k^j$. We still have $\|e_k^j y_0 e_k^j - y_{kj}\| \leq \varepsilon/4$, with $(1/3 - \varepsilon/12)e_k^j \leq y_{kj} \leq 1/2 e_k^j$ and $E_A(y_{kj}) = t_k e_k^j$. Let $p_k^j \in A(1 - e_k^j)$ be a projection of trace $\tau(e_k^j - y_{kj})/t_k$. To see that this is possible, we need to have $\tau(e_k^j - y_{kj})/t_k \leq \tau(1 - e_k^j)$, which is easily seen to hold true due to our choices. Note also that $\tau(p_k^j) \geq \tau(e_k^j)$. Take B_0 to be a separable diffuse abelian von Neumann subalgebra of $p_k^j M p_k^j$ which is perpendicular to $A p_k^j$. Let f_k^j be a projection in B_0 such that $\tau(f_k^j) = \tau(e_k^j)$. Let $v \in M$ be a partial isometry such that $v^* v = e_k^j$ and $v v^* = f_k^j$, which due to the assumption that any two separable diffuse abelian von Neumann subalgebras of $f_k^j M f_k^j$ are unitary conjugate, can be chosen so that in addition we have $v(e_k^j - y_{kj})v^* \in B_0$.

Denote $g_{kj} = y_{kj} + v(e_k^j - y_{kj})v^* + v(y_{kj}(e_k^j - y_{kj}))^{1/2} + (y_{kj}(e_k^j - y_{kj}))^{1/2}v^*$. It is easy to check that $g_{kj}^2 = g_{kj}$, i.e., $g_{kj} \in \mathcal{P}(M)$. Moreover

$$\begin{aligned} E_A(g_{kj}) &= E_A(p_k^j g_{kj} p_k^j + e_k^j g_{kj} e_k^j) = E_A(v(e_k^j - y_{kj})v^*) + E_A(y_{kj}) \\ &= \tau(v(e_k^j - y_{kj})v^*)/\tau(p_k^j) p_k^j + t_k e_k^j = t_k(p_k^j + e_k^j). \end{aligned}$$

Thus, by our assumption, each one of the projections $g_{kj} \in (p_k^j + e_k^j)M(p_k^j + e_k^j)$ can be paved over $A(p_k^j + e_k^j)$. Thus $y_{kj} = e_k^j g_{kj} e_k^j$ can be paved, so $\sum_{k,j} e_k^j y e_k^j$ can be paved, implying that y can be paved. By (2), it follows that y_0 can be paved, and by (1), x can be paved as well. Thus, any selfadjoint element can be paved, so by (3) any element in M can be paved.

Moreover, if we keep track of the number of projections necessary in the above pavings we see that in order to $\varepsilon/2$ -pave a selfadjoint element, $n(\mathcal{P}_0, \varepsilon/50)(\varepsilon^{-1} + 1)$ many projections are sufficient. By (3), we get that $n(\varepsilon) \leq n(\mathcal{P}_0, \varepsilon/50)^2(\varepsilon^{-1} + 1)^2$. \square

2.4. Remarks. 1° Let $A \subset M$ be a MASA in a II_1 factor and $x \in M \ominus A$, $\|x\| \leq 1$. If we view x as an element in M^ω , then its ε -paving number over A^ω is equal to n iff for any $\delta > 0$, there exists a partition of 1 with projections $p_1, \dots, p_n \in \mathcal{P}(A)$ with the property that the spectral projection of $|\sum_i p_i x p_i|$ corresponding to the interval (ε, ∞) has trace $\leq \delta$.

2° We already mentioned in 2.1 (b) several properties of the algebra \mathbf{R} : given any representation of R as crossed product $D \rtimes \Gamma$, for some free action of a (necessarily

countable amenable) group Γ on D , Γ acts freely on D^ω as well and we have $\mathbf{R} = D^\omega \rtimes \Gamma$; thus, \mathbf{R} is amenable and has D^ω as a Cartan subalgebra, but it has large, non-separable center. In addition, note that due to Rohlin's theorem and Følner's condition for amenable groups, any two free actions of the same amenable group $\Gamma \curvearrowright D^\omega$ are conjugate by a unitary element in $\mathcal{N}_{R^\omega}(D^\omega)$. Moreover, the 1-cohomology for $D^\omega \subset \mathbf{R}$ vanishes, so if $\Gamma, \Lambda \subset \mathcal{N}_R(D)$ are two countable amenable groups of unitaries that implement free actions on D and $\Delta : \Gamma \simeq \Lambda$ is a group isomorphism, then there exists $u \in \mathcal{N}_{R^\omega}(D^\omega)$ such that $uu_g u^* = \Delta(u_g)$, $\forall u_g \in \Gamma$. In particular, if $u_1, u_2 \in \mathcal{N}_R(D)$ act freely on D , then there exists $u \in \mathcal{N}_{R^\omega}(D^\omega)$ such that $uu_1 u^* = u_2$. Note that in fact all these properties hold also for countable amenable subgroups $\Gamma \subset \mathcal{N}_M(\mathbf{D})$ acting freely on \mathbf{D} .

3° Recall that in (4.1.(iii) and 4.3.3° of [P10]) it was shown that if B is a separable amenable von Neumann subalgebra in a II_1 von Neumann algebra M such that the Pimsner-Popa index [PiP] of the inclusions $pBp \subset pMp$ is infinite for any projection $p \in B$, $p \neq 0$, then there exist non-normal conditional expectations of M onto B , and thus E_B is not the unique conditional expectation of M onto B . In particular, if A is a separable MASA in a II_1 von Neumann algebra M , then E_A is not the unique conditional expectation of M onto A , and thus $A \subset M$ cannot have the KS property, nor the paving property. We recall the argument in [P10], emphasizing a simplification that occurs in the case of a MASA.

First one constructs a singular state φ on M with $\varphi|_B = \tau$ as follows: Like in [P10], the hypothesis implies $\exists b_n \in L^1 M_+$ with $\tau(s(b_n)) \leq 2^{-n}$, $E_B(b_n) = 1$. (In the case $B = A$ is a MASA in M , the argument becomes much simpler, as one can take $b_n = 2^n q_n \in M$, with q_n the following projection: let $e_{kk} \in A$ be a partition of 1 with 2^n mutually equivalent projections and complement it to a set $e_{jk} \in M$ of matrix units, then define $q_n = 2^{-n} \sum_{j,k} e_{jk}$, which is a projection in M with $E_A(q_n) = \sum_k e_{kk} E_A(q_n) e_{kk} = E_A(\sum_k e_{kk} q_n e_{kk}) = 2^{-n}$.) Then $\varphi_n = \tau(\cdot b_n)$ defines a normal state on M which, since $E_B(b_n) = 1$, satisfies $\tau(y b_n) = \tau(y)$, $\forall y \in B$. Take φ to be a state on M obtained as a weak-limit of φ_n . Then we still have $\varphi|_B = \tau|_B$ while φ is singular on M (because for each fixed n we have $\varphi(1 - \vee_{m \geq n} s(b_m)) = 0$ and $\lim_n (1 - \vee_{m \geq n} s(b_m)) = 1$).

Next, since B is amenable, by Connes' Theorem we can find a countable amenable subgroup $\mathcal{U}_0 \subset \mathcal{U}(B)$ such that $\mathcal{U}_0'' = B$. For each $x \in M$, put $\psi(x) = \int \varphi(uxu^*) du$, where the integral is in the Banach limit sense, over an invariant mean on the countable amenable group \mathcal{U}_0 . Then ψ defines a state on M which is in the $\sigma(M^*, M)$ -closure of a countable set of singular states on M . By [Ak], it follows that ψ is singular as well. Also, by its definition, ψ has the span of \mathcal{U}_0 in its centralizer and $\psi|_B = \tau|_B$. If now $a \in B$ is arbitrary and $a_n \in \text{sp} \mathcal{U}_0$, $\|a_n\| \leq \|x\|$

are so that $\|x - b_n\|_2 \rightarrow 0$, then by Cauchy-Schwartz inequality for ψ , for all $x \in M$ we have

$$\begin{aligned} |\psi(ax) - \psi(a_nx)| &\leq \psi((a - a_n)(a - a_n))^1/2 \psi(x^*x)^1/2 \\ &= \tau((a - a_n)(a - a_n)^*)^1/2 \psi(x^*x)^1/2 = \|a - a_n\|_2 \psi(x^*x)^1/2 \rightarrow 0. \end{aligned}$$

Similarly, $|\psi(xa) - \psi(xa_n)| \rightarrow 0$. Since $\psi(a_nx) = \psi(xa_n)$, $\forall n$, this shows that $\psi(ax) = \psi(xa)$, i.e. B is in the centralizer of ψ . Taking $E : M \rightarrow B$ to be the unique conditional expectation satisfying $\psi(E(x)a) = \psi(xa)$, $\forall x \in M, a \in B$, we have constructed this way a singular (thus non-normal) conditional expectation of M onto B .

3. PAVING IN THE L^2 -NORM

Given a MASA A in a finite von Neumann algebra M and x an element in M with $E_A(x) = 0$, our strategy for estimating the norm of its paving $y = \sum_i p_i x p_i$, for $p_i \in \mathcal{P}(A)$ a partition of 1, will be to calculate the moments $\tau((y^*y)^n)$ and use the well known formula $\|y\|^2 = \lim_n \tau((y^*y)^n)^{1/n}$. More generally, in order to have $\|y\| \leq c$, we need to prove that $\tau((y^*y)^n) \leq c^{2n}$, for large enough n . One way of controlling these moments is to construct the partitions $\{p_i\}_i \subset A$ so that to have “high order of independence” with respect to x .

We will use the following terminology in this respect: Two sets $V, W \subset M \ominus \mathbb{C}$ are *n-independent* if any alternating word $x_1 y_1 \dots x_k y_k$, with $k \leq n$ and $x_1 \in V \cup \{1\}$, $x_2, \dots, x_k \in V$, $y_1, \dots, y_{k-1} \in W$, $y_k \in W \cup \{1\}$, has trace 0. An algebra $B_0 \subset M$ is *n-independent* to V if V and $B_0 \ominus \mathbb{C}$ are *n-independent*. Note that 1-independence amounts to what one usually calls τ -independence. In our work, independence will in fact also appear (more or less implicitly) relative to subalgebras. The “generic” meaning of such relative independence is the following: if a von Neumann subalgebra $B_1 \subset M$ is given, then V, W are *n-independent relative to B_1* , if any alternating word with at most $2n$ letters in V, W has 0-expectation onto B_1 .

In this section we prove a general fact along these lines, showing that given any MASA A in a finite von Neumann algebra M , we can find partitions of 1 in A that are “asymptotically 2-independent” with respect to any given countable set of elements in $M \ominus A$. In other words, given any countable set $X \subset M \ominus A$, there exists a diffuse abelian subalgebra $B_0 \subset A^\omega$ such that any word with at most 4 alternating letters in X and $B_0 \ominus \mathbb{C}$ has trace 0. In particular, $\|p_i x p_j\|_2 = \|p_i\|_2 \|p_j\|_2 \|x\|_2$, for any $p_i, p_j \in \mathcal{P}(A)$, $x \in X$, so any partition $p_i \in B_0$ with small mesh gives L^2 -pavings of $x \in X$, simultaneously for all $x \in X$: $\|\sum_i p_i x p_i\|_2^2 = \|x\|_2^2 \sum_i \tau(p_i)^2 \leq \max_i \{\tau(p_i)\}_i \|x\|_2^2$. We construct such partitions recursively, but another method, where moments are controlled through incremental patching, can be used instead (see Section 5.3).

3.1. Lemma. 1° If $\xi \in L^2M$, $u \in \mathcal{U}(M)$ are so that $u^2 = 1$, $|\tau(\xi^*u\xi u^*)| \leq c\|\xi\|_2^2$, for some $1 \geq c > 0$, then $p_1 = (1+u)/2$, $p_2 = (1-u)/2$ is a partition of 1 with projections satisfying $\|p_1\xi p_1 + p_2\xi p_2\|_2 \leq (1+c)/\sqrt{2}\|\xi\|_2$. If in addition $|\tau(u\xi^*\xi)| \leq c\|\xi\|_2^2$ and $|\tau(u)| \leq c/2$, then $\|p_i\xi p_i\|_2 \leq (1+2c)/\sqrt{2}\|\xi\|_2\|p_i\|_2$, $i = 1, 2$.

2° If $\xi \in L^2M$ and $u \in \mathcal{U}(M)$ satisfy $\tau(\xi^*u\xi u^*) \leq c\|\xi\|_2^2$, for some $c \leq 2^{-7}$, and $n \geq 2^7$, then the spectral projections $\{e_k\}_{1 \leq k \leq n}$ of u defined by $e_k = e_{[e^{2\pi i(k-1)/n}, e^{2\pi i k/n})}(u)$, give a partition of 1 and satisfy $\|\sum_k e_k \xi e_k\|_2 \leq 3/4\|\xi\|_2$.

Proof. 1° We have $\|\xi + u\xi u^*\|_2^2 = 2\|\xi\|_2^2 + 2\Re\tau(\xi^*u\xi u^*) \leq (2+2c)\|\xi\|_2^2$. Since $p_1\xi p_1 + p_2\xi p_2 = 2^{-1}(\xi + u\xi u^*)$, we get $\|p_1\xi p_1 + p_2\xi p_2\|_2^2 \leq (1+c)/2\|\xi\|_2^2$ and the first part of the statement follows. If $|\tau(u\xi^*\xi)| \leq c\|\xi\|_2^2$ and $|\tau(u)| \leq c/2$ as well, then $|\tau(p_1) - 1/2| = |\tau(u)|/2 \leq c/4$. Thus $\tau(p_1) \geq 1/2 - c/4 \geq 1/4$ and also $1/2 \leq \tau(p_1) + c/4 \leq \tau(p_1)(1+c)/4$, so we get:

$$\begin{aligned} \|p_1\xi p_1\|_2^2 &= \tau((1+u)\xi^*(1+u)\xi)/4 = \tau(\xi^*\xi)/4 + \tau(u\xi^*\xi)/2 + \tau(u\xi^*u\xi)/4 \\ &\leq (1/4 + 3c/4)\|\xi\|_2^2 \leq (1+3c)/2\|\xi\|_2^2(1+c)\tau(p_1) \leq (1+3c)^2/2\|\xi\|_2^2\|p_1\|_2^2. \end{aligned}$$

Similarly, by using that $p_2 = (1-u)/2$, we obtain

$$\|p_2\xi p_2\|_2^2 \leq (1+3c)/2\|\xi\|_2^2(1+c)\tau(p_2) \leq (1+2c)^2/2\|\xi\|_2^2\|p_2\|_2^2.$$

2° We may clearly assume $\|\xi\|_2 = 1$. Note that if we denote $\lambda_k = e^{2\pi i k/n}$, then $\|u - \sum_k \lambda_k e_k\| \leq |2\pi i/n - 1| < 2\pi/n$. Since the elements $\{e_j \xi e_k\}_{1 \leq j, k \leq n}$ are mutually orthogonal in the Hilbert space L^2M , by using first Pythagora's theorem and then the inequality $|\lambda_j^* \lambda_k - 1| \leq 2, \forall j, k$, we get:

$$\begin{aligned} 4 - 4\|\sum_k e_k \xi e_k\|_2^2 &= 4\|\xi\|_2^2 - 4\|\sum_k e_k \xi e_k\|_2^2 = 4\|\sum_{j \neq k} e_j \xi e_k\|_2^2 \\ &\geq \|\sum_{j \neq k} (\lambda_j^* \lambda_k - 1) e_j \xi e_k\|_2^2 = \|(\sum_j \lambda_j^* e_j) \xi (\sum_k \lambda_k e_k) - \xi\|_2^2 \\ &\geq \|u^* \xi u - \xi\|_2^2 - 4\|u - \sum_k \lambda_k e_k\| \\ &= 2 - 2\Re\tau(\xi^* u^* \xi u) - 4\|u - \sum_k \lambda_k e_k\| \geq 2 - 2c - 8\pi/n. \end{aligned}$$

If we now choose $c < 2^{-7}$ and $n \geq 2^7$, then from the first and last term of the above estimates we get

$$\|\sum_k e_k \xi e_k\|_2^2 \leq 1/2 + c/2 + 2\pi/n \leq 9/16.$$

□

3.2. Remark. For the following lemmas, it will be useful to recall that a unitary representation π of a group G on a Hilbert space \mathcal{H} is *weak mixing* if any of the following equivalent conditions is satisfied:

(3.2.1) Given any finite subset $F \subset \mathcal{H}$ and any $\varepsilon > 0$, there exists $g \in G$ such that $|\langle \pi(g)(\xi), \eta \rangle| \leq \varepsilon$, $\forall \xi, \eta \in F$;

(3.2.2) π has no non-zero finite dimensional invariant subspaces $\mathcal{H}_0 \subset \mathcal{H}$;

(3.2.3) The representation $\pi \otimes \bar{\pi}$ of G on $\mathcal{H} \otimes \bar{\mathcal{H}}$ is ergodic, i.e. it has no fixed non-zero vectors.

(3.2.4) For any finite dimensional subspace $\mathcal{H}_0 \subset \mathcal{H}$ and any $\varepsilon > 0$, there exists $g \in G$ such that $\pi(g)(\mathcal{H}_0) \perp_{\varepsilon} \mathcal{H}_0$, i.e. $|\langle \pi(g)(\xi), \eta \rangle| \leq \|\xi\| \|\eta\|$, $\forall \xi, \eta \in \mathcal{H}_0$.

3.3. Lemma. *If $B \subset M$ is a diffuse von Neumann subalgebra, then the action Ad of its unitary group $\mathcal{U}(B)$ on $L^2(M \ominus (B' \cap M))$ is weak mixing. Moreover, if B is abelian, then the restriction of this Ad -action to the subgroup of period 2 elements, $\mathcal{U}^0(B) \stackrel{\text{def}}{=} \{u \in \mathcal{U}(B) \mid u^2 = 1\}$, is still weak mixing.*

Proof. If the action is not weak mixing, then there exists a non-zero finite dimensional subspace $\mathcal{H}_0 \subset L^2(M \ominus (B' \cap M))$ satisfying $u\mathcal{H}_0u^* = \mathcal{H}_0$, $\forall u \in \mathcal{U}(B)$. In particular, if $A \subset B$ is a diffuse abelian subalgebra of B and $\mathcal{U}^0 = \mathcal{U}^0(A)$ denotes the group of unitaries of period two in A as in part 2°, then \mathcal{H}_0 is invariant to the representation $\xi \mapsto \text{Ad}(u)(\xi) = u\xi u^*$ of \mathcal{U}^0 on \mathcal{H}_0 . Since the image of this representation is an abelian subgroup \mathcal{V}^0 of $\mathcal{U}(\mathcal{H}_0)$, it can be diagonalized. Thus, $\mathcal{H}_0 = \sum_j \mathbb{C}\xi_j$, with $\xi_1, \xi_2, \dots, \xi_n$ an orthonormal basis of \mathcal{H}_0 such that $\text{Ad}(\mathcal{U}^0)(\xi_j) \subset \mathbb{T}\xi_j$, $\forall j$, and since any element in \mathcal{V}^0 has period 2, we actually have $\text{Ad}(\mathcal{U}^0)(\xi_j) \subset \{\pm\xi_j\}$, $\forall j$. But the group $(\mathcal{U}^0, \|\cdot\|_2)$ is Polish and contractible. This can be seen by taking a $\|\cdot\|_2$ -continuous path $\{p_t \mid 0 \leq t \leq 1\} \subset \mathcal{P}(A)$ with $\tau(p_t) = t$, $p_t \leq p_{t'}$ iff $t \leq t'$, then defining the continuous path of group morphisms $\Phi_t : \mathcal{U}^0(A) \rightarrow \mathcal{U}^0(A)$, by $\Phi_t(u) = p_t + u(1 - p_t)$, which satisfies $\Phi_0(u) = u$, $\Phi_1(u) = 1$, $\forall u \in \mathcal{U}^0(A)$. Since \mathcal{U}^0 is contractible and the representation Ad is continuous, and since the one dimensional representation $u \mapsto \text{Ad}(u)$ lies in $\{\pm 1\}$, this representation must be trivial, i.e. $u\xi_j u^* = \xi_j$, $\forall u \in \mathcal{U}^0$, $\forall j$. Hence, $u\xi = \xi u$ for all $\xi \in \mathcal{H}_0$ and for all $u \in \mathcal{U}(A)$ (because \mathcal{U}^0 generates A as a von Neumann algebra). Since any $u \in \mathcal{U}(B)$ lies in a diffuse abelian von Neumann subalgebra $A \subset B$, it follows that $u\xi = \xi u$, for all $u \in \mathcal{U}(B)$ and all $\xi \in \mathcal{H}_0$. This means that $\mathcal{H}_0 \subset L^2(B' \cap M)$, while at the same time $\mathcal{H}_0 \perp B' \cap M$, implying that $\mathcal{H}_0 = 0$. □

We have seen in the previous lemma that if A is a diffuse abelian von Neumann subalgebra in M , then the Ad -action of the group $\mathcal{U}^0(A)$ (of period 2 unitaries in A) on $L^2(M \ominus A' \cap M)$ is weak mixing. We will show in the next lemma

that one can choose the corresponding mixing elements in $\mathcal{U}^0(A)$ so that to be τ -independent with respect to any given finite set in M and to have 0-trace. Proving this in the abelian case is quite straightforward. But due to its possible independent interest, we will actually prove this type of result for arbitrary diffuse von Neumann subalgebras $B \subset M$, a fact that will make the argument a bit more lengthy.

3.4. Lemma. *Let M be a finite von Neumann algebra and $B \subset M$ a diffuse von Neumann subalgebra. Given any finite dimensional subspaces $X \subset L^2(M \ominus B \vee (B' \cap M))$, $Y \subset L^1 M$, and any $\delta > 0$, there exists a period 2 unitary element $u \in B$ such that $|\tau(u\xi_1^* u^* \xi_2)| \leq \delta \|\xi_1\|_2 \|\xi_2\|_2$, $|\tau(u\eta)| \leq \delta \|\eta\|_1$, $\forall \xi \in X$, $\eta \in Y$.*

Proof. We first prove that given any finite dimensional subspace $\mathcal{H}_0 \subset L^2(M \ominus B \vee (B' \cap M))$, there exists a diffuse abelian von Neumann subalgebra $A \subset B$ such that $E_{A' \cap M}(\mathcal{H}_0) = 0$. Since \mathcal{H}_0 is perpendicular to $B \vee (B' \cap M)$, it is also perpendicular to $B' \cap M$, so by Lemma 3.3 there exists $u \in \mathcal{U}(B)$ such that $|\tau(\xi^* u \xi u^*)| \leq c \|\xi\|_2^2$, $\forall \xi \in \mathcal{H}_0$. By Lemma 3.1, if $c = 2^{-7}$ and $n_1 = 2^7$, then there exists a partition of 1 with n_1 projections $\{e_j^1\}_j$ in B such that $\|\sum_j e_j^1 \xi e_j^1\|_2 \leq 3/4 \|\xi\|_2$, $\xi \in \mathcal{H}_0$. Since $\mathcal{H}_0 \perp B \vee B' \cap M$, we also have $\sum_j e_j^1 \mathcal{H}_0 e_j^1 \perp \sum_j e_j^1 (B \vee B' \cap M) e_j^1 = B_1 \vee (B_1' \cap M)$, where $B_1 = \sum_j e_j^1 B e_j^1$ (cf. Lemma 2.1 in [P1]). We can thus continue recursively, replacing \mathcal{H}_0 by $\sum_j e_j^1 \mathcal{H}_0 e_j^1$ and B by B_1 , to get a partition of 1 with n_2 projections $\{e_k^2\}_k \subset B_1$, which refines $\{e_j^1\}_j$ and satisfies $\|\sum_j e_k^2 \xi e_k^2\|_2 \leq 3/4 \|\sum_k e_k^1 \xi e_k^1\|_2$. Thus,

$$\|\sum_k e_k^2 \xi e_k^2\|_2 \leq 3/4 \|\sum_j e_j^1 \xi e_j^1\|_2 \leq (3/4)^2 \|\sum_k e_k^2 \xi e_k^2\|_2, \forall \xi \in \mathcal{H}_0.$$

By iterating this procedure we get a sequence of finer and finer partitions of 1, $\{e_j^m\}_{j=1}^{n_m} \subset B$, where $n_m = 2^{7m}$, such that $\|\sum_i e_j^m \xi e_j^m\|_2 \leq (3/4)^m \|\xi\|_2 \leq \varepsilon \|\xi\|_2$, $\forall \xi \in \mathcal{H}_0$ (Note that the number $n = n_m$ of projections necessary to get $(3/4)^m \leq \varepsilon$ is majorized by $2^{7 \ln(1/\varepsilon) / \ln(4/3) + 1}$, thus $n \leq 2^7 (1/\varepsilon)^{7 \ln 2 / \ln(4/3)} \approx 2^7 (1/\varepsilon)^{17.02}$, so the order of magnitude of the size of the partition is $\varepsilon^{-17.02}$).

If we define A to be the von Neumann algebra generated by $\{e_j^m \mid 1 \leq j \leq n_m, m \geq 1\}$, it follows that $E_{A' \cap M}(\xi) = 0$, $\forall \xi \in \mathcal{H}_0$. Consider now the group $\mathcal{U}^0 = \mathcal{U}^0(A)$ and note that it is Polish with respect to the topology implemented by $\|\cdot\|_2$. Also, $(\mathcal{U}^0, \|\cdot\|_2)$ is connected, in fact even contractible (due to the same construction as in the above proof of Lemma 3.3). Consider the Hilbert space $\mathcal{K} = HS(L^2(\overline{\text{sp}}(AFA))) \oplus HS(L^2 M)$ and the unitary representation π of \mathcal{U}^0 on \mathcal{K} given by $u \mapsto \text{Ad}(L_u R_u) \oplus \text{Ad}(L_u)$, $\forall u \in \mathcal{U}^0$, where $HS(\mathcal{H})$ denotes the space of Hilbert-Schmidt operators on the Hilbert space \mathcal{H} (i.e. $HS(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) \mid \text{Tr}(x^* x) < \infty\}$), and L_u (resp. R_u) are the operators of left (resp. right) multiplication by $u \in \mathcal{U}^0 \subset A$. Thus, if we identify in the usual way $HS(\mathcal{H}) \simeq \mathcal{H} \overline{\otimes} \mathcal{H}^*$, then for each $\xi_1, \xi_2 \in \overline{\text{sp}}(A\mathcal{H}_0 A)$, $\eta_1, \eta_2 \in L^2 M$ we have

$$\pi(u)(\xi_1 \otimes \xi_2^* \oplus \eta_1 \otimes \eta_2^*) = (u\xi_1 u^* \otimes u\xi_2^* u^* \oplus u\eta_1 \otimes \eta_2^* u^*).$$

Let $p = p_1 \oplus p_2 \in \mathcal{K}$, where p_1 is the orthogonal projection of $\overline{\text{sp}}(A\mathcal{H}_0 A)$ onto \mathcal{H}_0 and p_2 is the orthogonal projection of $L^2 M$ onto this same space. Let $K_p = \overline{\text{co}}^w \{ \pi(u)(p) \mid u \in \mathcal{U}^0 \}$. Note that $\pi(u)(K_p) = K_p$ and $\|\pi(u)(x)\|_{\mathcal{K}} = \|x\|_{\mathcal{K}}$, $\forall x \in \mathcal{K}$. Note also that all elements x in K_p are of the form $x = x_1 \oplus x_2$, with $x_1 \in HS(\overline{\text{sp}}(AFA)) \subset \mathcal{B}(\overline{\text{sp}}(AFA))$, $x_2 \in HS(L^2 M) \subset \mathcal{B}(L^2 M)$ positive operators when viewed as acting on the corresponding Hilbert space.

Since K_p is convex, weakly closed and bounded in \mathcal{K} , there exists a unique element $b = b_1 \oplus b_2 \in K_p$ of minimal Hilbert norm $\|\cdot\|_{\mathcal{K}}$. Since $\pi(u)(b) \in K_p$ and has the same norm as b , it follows that $\pi(u)(b) = b$, $\forall u \in \mathcal{U}_0$. Thus, $\text{Ad}(L_u R_u)(b_1) = b_1$, $\text{Ad}(L_u)(b_2) = b_2$, $\forall u \in \mathcal{U}^0$. This means that, as (positive) operators on the corresponding Hilbert space, b_1 commutes with $L_u R_u$ and b_2 with L_u , $\forall u \in \mathcal{U}^0$. Thus, the spectral decomposition of b_1 (resp b_2) commutes with these unitaries. Since b_1, b_2 are Hilbert-Schmidt, they are in particular compact, so any spectral projection corresponding to (t, ∞) for $t > 0$ is finite dimensional. It follows that if $b_1 \neq 0$ (resp. $b_2 \neq 0$) then there exists a non-zero finite dimensional subspace $\mathcal{H}_1 \subset \overline{\text{sp}}(AFA)$ such that $u\mathcal{H}_1 u^* = \mathcal{H}_1$ (resp. $\mathcal{H}_2 \subset L^2 M$ such that $u\mathcal{H}_2 = \mathcal{H}_2$), $\forall u \in \mathcal{H}_2$.

Let us first notice that this implies $\mathcal{H}_2 = 0$. Indeed, because $\mathcal{U}^0 \mathcal{H}_2 = \mathcal{H}_2$ implies $A\mathcal{H}_2 = \mathcal{H}_2$, which contradicts the finite dimensionality of \mathcal{H}_2 , unless $\mathcal{H}_2 = 0$. To see that $\mathcal{H}_1 = 0$ as well, note that $\mathcal{H}_1 \ni \xi \mapsto \text{Ad}(u)(\xi) \in \mathcal{H}_1$, $\forall u \in \mathcal{U}^0$ defines a continuous unitary representation of the abelian Polish group \mathcal{U}^0 on \mathcal{H}_1 . Since the image of this representation is an abelian subgroup of $\mathcal{U}(\mathcal{H}_0)$, it can be diagonalized. Thus, $\mathcal{H}_0 = \Sigma_j \mathbb{C} \xi_j$, with $\xi_1, \xi_2, \dots, \xi_n$ an orthonormal basis of \mathcal{H}_0 such that $\text{Ad}(\mathcal{U}^0)(\xi_j) \subset \mathbb{T} \xi_j$, $\forall j$. Since any element in \mathcal{U}^0 has period 2, we actually have $\text{Ad}(\mathcal{U}^0)(\xi_j) \subset \{\pm \xi_j\}$, $\forall j$. But as we have noticed above, the Polish group $(\mathcal{U}^0, \|\cdot\|_2)$ is connected (even contractible), implying that $u\xi_j u^* = \xi_j$, or equivalently $u\xi = \xi u$, $\forall u \in \mathcal{U}^0$, $\forall j$. Hence, $a\xi = \xi a$ for all $\xi \in \mathcal{H}_1$ and all $a \in A$ (because \mathcal{U}^0 generates A as a von Neumann algebra), i.e. $\mathcal{H}_1 \subset L^2(A' \cap M)$. But $E_{A' \cap M}(F) = 0$ implies $E_{A' \cap M}(AFA) = 0$ and thus $E_{A' \cap M}(\overline{\text{sp}}(AFA)) = 0$, so in particular $E_{A' \cap M}(\mathcal{H}_1) = 0$. We have thus shown that $\mathcal{H}_1 \subset L^2(A' \cap M)$ and $\mathcal{H}_1 \perp L^2(A' \cap M)$, showing that $\mathcal{H}_1 = 0$.

This implies $b = 0$ and thus $0 \in K_p$. Hence, for any $\delta > 0$ there exists $u \in \mathcal{U}^0$ such that $\text{Tr}(\pi(u)(p)p) < \delta$. Indeed, for if there exists $\delta_0 > 0$ such that $\text{Tr}(\pi(u)(p)p) \geq \delta_0$, $\forall u \in \mathcal{U}^0$, then $\text{Tr}(xp) \geq \delta_0$ for all $x \in K_p$, in particular for $x = 0 \in K_p$, giving $0 \geq \delta_0$, a contradiction.

But $\text{Tr}(\pi(u)(p)p) < \delta$ implies that we have both $u\mathcal{H}_0 \perp_{\delta} \mathcal{H}_0$ and $u\mathcal{H}_0 u^* \perp_{\delta} \mathcal{H}_0$, in particular $|\tau(u\xi_1^* u^* \xi_2)| < \delta \|\xi_1\|_2 \|\xi_2\|_2$, $|\tau(u\eta)| < \delta \|\eta\|_1$, for all non-zero elements

$\xi_1, \xi_2 \in X$, $\eta \in Y'Y'^* + \mathbb{C}1$. Thus, if we embed $Y(\subset L^1M)$ in some $Y'Y'^*$ for some appropriate finite subspace $Y' \subset L^2M$, then all required conditions are satisfied. \square

3.5. Lemma. *Let M be a finite von Neumann algebra and $\mathcal{H}_1 \subset L^2M$, $\mathcal{H}_2 \subset L^1M$ finite dimensional spaces. Given any $\delta > 0$, there exists $\delta' > 0$ such that if $x \in M$ satisfies $\|x\| \leq 1$ and $\|x\|_2 \leq \delta'$, then $\|x\xi\|_2 \leq \delta\|\xi\|_2$, $\|x\eta\|_1 \leq \delta\|\eta\|_1$, $\forall \xi \in \mathcal{H}_1$, $\eta \in \mathcal{H}_2$.*

Proof. The first part follows from the fact that norm $\|\cdot\|_2$ implements the so -topology on the unit ball of M while the product with a compact operator (such as the orthogonal projection of L^2M onto the finite dimensional space \mathcal{H}_1) turns so -convergence into operator norm convergence. To prove the second part, note that it is sufficient to show that given any $\delta > 0$ and any finite set $\{\eta_i\}_i \subset \mathcal{H}_2$ which is “ $\delta/2$ -dense” in the L^1 -unit ball of \mathcal{H}_2 , there exists $\delta' > 0$ such that if $x \in M$ satisfies $\|x\| \leq 1$, $\|x\|_2 \leq \delta'$, then $\|x\eta_i\|_1 \leq \delta/2$. In turn, this fact is an immediate consequence of the first part, the Cauchy-Schwartz inequality and the fact that any $\eta \in L^1M$, $\|\eta\|_1 = 1$ can be decomposed as a product $\xi_1\xi_2$ with $\xi_i \in L^2M$, $\|\xi_1\|_2 = \|\xi_2\|_2 = 1$. \square

3.6. Theorem. *Let $n \geq 1$ be an integer. Given any finite von Neumann algebra M , any diffuse von Neumann subalgebra $B \subset M$, any finite sets $X \subset L^2(M \ominus B \vee (B' \cap M))$, $Y \subset L^1M$ and any $\alpha > 0$, there exists a finite dimensional von Neumann subalgebra $C \subset B$ generated by 2^n minimal projections of trace 2^{-n} such that*

- (a) $|\tau(a_1\xi_1a_2\xi_2)| \leq \alpha\|a_1\|_2\|a_2\|_2$, $\forall \xi_1, \xi_2 \in X$, $\forall a_1, a_2 \in C \ominus \mathbb{C}$.
- (b) $|\tau(\eta a)| \leq \alpha\|a\|$, $\forall a \in C \ominus \mathbb{C}$, $\forall \eta \in Y \cup XX^*$.

In particular, if $q_1, \dots, q_{2^n} \in C$ are the minimal projections in C , then

- (a') $|||q_i\xi q_j|||_2^2 - \|\xi\|_2^2|\tau(q_i)\tau(q_j)| \leq 3 \cdot 2^{-n}\alpha$, $\forall i, j$, $\forall \xi \in X$;
- (b') $|\tau(\eta q_i) - \tau(\eta)\tau(q_i)| \leq \alpha$, $\forall i$, $\forall \eta \in Y \cup XX^*$.
- (c') $\|q_i\xi q_i\|_2 \leq (2^{-n/2}\|\xi\|_2 + 2\alpha^{1/2})\|q_i\|_2$; $\|\sum_i q_i\xi q_i\|_2^2 \leq 2^{-n}\|\xi\|_2^2 + 3\alpha$, $\forall i$, $\forall \xi \in X$.
- (d') $\|q_i\xi q_i\|_1 \leq (2^{-n/2}\|\xi\|_2 + 2\alpha^{1/2})\tau(q_i)$, $\forall i$, $\forall \xi \in X$.

Proof. Note that, without any loss of generality, we may assume $X = X^*$, $Y = Y^*$, $\|\xi\|_2 = 1$, $\|\eta\|_1 = 1$, $\forall \xi \in X$, $\forall \eta \in Y$.

We prove the statement by induction over $n \geq 0$. If $n = 0$ then $C = \mathbb{C}1$ and the conditions are trivially satisfied. Assume we have proved the statement up to some n . Thus, there exists an abelian 2^n -dimensional $*$ -subalgebra $C^0 \subset B$ generated by

minimal projections $q_1^0, \dots, q_{2^n}^0 \in B$ of trace 2^{-n} such that for all $a, a_1, a_2 \in C^0 \ominus \mathbb{C}$, $\xi_1, \xi_2 \in X$, $\eta \in Y \cup XX^* \cup \{1\}$ we have

$$(1) \quad |\tau(a_1 \xi_1 a_2 \xi_2)| \leq \alpha' \|a_1\|_2 \|a_2\|_2, |\tau(\eta a)| \leq \alpha' \|a\|,$$

where $\alpha' = 2^{-n-2}\alpha$.

Denote $B_0 = \Sigma_i q_i B q_i$ and let X_0 , respectively Y_0 , be the linear span of the finite set $\Sigma_{i,j} q_i^0 X q_j^0$, respectively $Y \cup X_0 X_0^* \cup \{q_i^0 \mid 1 \leq i \leq 2^n\}$. Note that the condition $X \perp B \vee B' \cap M$ implies $X_0 \perp B_0 \vee B'_0 \cap M$. Indeed, by Lemma 2.1 in [P1], we have $B_0 \vee B'_0 \cap M = \Sigma_i q_i^0 (B \vee B' \cap M) q_i^0$ and so if $x \in X$, $y \in B \vee B' \cap M$, then

$$\tau(q_j^0 x q_k^0 \Sigma_i q_i^0 y q_i^0) = \delta_{jk} \tau(q_j^0 x q_j^0 y q_j^0) = \delta_{jk} \tau(x q_j^0 y q_j^0) = 0,$$

the latter equality being due to the fact that $q_j^0 y q_j^0 \in B \vee B' \cap M$.

Let $\delta = 2^{-n-2}\alpha$. By Lemma 3.5, there exists $1 \geq \delta' > 0$ such that if $x \in M$ satisfies $\|x\| \leq 1$ and $\|x\|_2 \leq \delta'$, then $\|x\xi\|_2 \leq 3^{-1}\delta\|\xi\|_2$, $\|x\eta\|_1 \leq 3^{-1}\delta\|\eta\|_1$, $\forall \xi \in X_0, \eta \in Y_0$. By Lemma 3.4, there exists $v \in \mathcal{U}^0(B_0)$ such that

$$(2) \quad |\tau(v \xi_1^* v^* \xi_2)| \leq 3^{-1}\delta\|\xi_1\|_2\|\xi_2\|_2, |\tau(\eta v)| \leq \delta'^2\|\eta\|_1, \forall \xi_1, \xi_2 \in X_0, \eta \in Y_0.$$

Since v is a period 2 unitary commuting with all q_i^0 , and $|\tau(v q_i^0)| \leq 2^{-n}\delta'^2$ (by last part of (2) and the fact that $q_i^0 \in Y_0$), using the fact that B_0 is diffuse we can split each projection q_i^0 into the sum of two projections $q_i^0 = q_{2i-1} + q_{2i}$ of trace 2^{-n-1} such that the period 2 unitary element $u = \Sigma_i (q_{2i-1} - q_{2i})$ satisfies $\|u - v\|_2 \leq \delta'$. Thus, u satisfies $\|(v - u)\xi\|_2 \leq 3^{-1}\delta\|\xi\|_2$, $\|(v - u)\eta\|_1 \leq 3^{-1}\delta\|\eta\|_1$, $\forall \xi \in X_0, \eta \in Y_0$. Combining with (2) and using the Cauchy-Schwartz inequality, we get:

$$\begin{aligned} (3) \quad |\tau(u \xi_1^* u^* \xi_2)| &\leq |\tau(v \xi_1^* v^* \xi_2)| + \|(u - v)\xi_1^*\|_2 \|u^* \xi_2\|_2 \\ &\leq |\tau(v \xi_1^* v^* \xi_2)| + \|v \xi_1^*\|_2 \|(u - v)\xi_2\|_2 + \|(u - v)\xi_1^*\|_2 \|u^* \xi_2\|_2 \\ &\leq |\tau(v \xi_1^* v^* \xi_2)| + 2\delta\|\xi_1\|_2\|\xi_2\|_2/3 \leq \delta\|\xi_1\|_2\|\xi_2\|_2. \end{aligned}$$

Moreover, since $\delta' \leq 1/3$ we have $\delta'^2 \leq \delta/3'$ and thus

$$(3') \quad |\tau(u\eta)| \leq |\tau((u - v)\eta)| + |\tau(v\eta)| \leq \delta\|\eta\|_1, \forall \eta \in Y_0.$$

Denote C the linear span of $\{q_j \mid 1 \leq j \leq 2^{n+1}\}$ and note that $C = C^0 + uC^0$, $C^0 \perp uC^0$. Let $x_i = a_i + ub_i \in C$, with $a_i, b_i \in C^0$. Thus, $\|x_i\|_2^2 = \|a_i\|_2^2 + \|b_i\|_2^2$, $i = 1, 2$. Take $\xi_1, \xi_2 \in X$. Then we have

$$\begin{aligned}
 (4) \quad & |\tau(x_1\xi_1x_2\xi_2)| \\
 &= |\tau(a_1\xi_1a_2\xi_2)| + |\tau(b_1u\xi_1a_2\xi_2)| + |\tau(a_1\xi_1ub_2\xi_2)| + |\tau(b_1u\xi_1b_2u\xi_2)| \\
 &= |\tau(a_1\xi_1a_2\xi_2)| + |\tau(u(\xi_1a_2\xi_2b_1))| + |\tau(u(b_2\xi_2a_1\xi_1))| + |\tau(u(\xi_1b_2)u(\xi_2b_1))|
 \end{aligned}$$

By (1), for the first term on the last line in (4), we have the estimate

$$(5) \quad |\tau(a_1\xi_1a_2\xi_2)| \leq \alpha' \|a_1\|_2 \|a_2\|_2 \leq \alpha' \|x_1\|_2 \|x_2\|_2.$$

Since $\xi_1a_2\xi_2b_1$ and $b_2\xi_2a_1\xi_1$ belong to Y_0 , by (3') it follows that for the second term on the last line of (4) we have the estimate:

$$\begin{aligned}
 (6) \quad & |\tau(u(\xi_1a_2\xi_2b_1))| \leq \alpha' \|\xi_1a_2\xi_2b_1\|_1 \leq \alpha' \|\xi_1a_2\|_2 \|\xi_2b_1\|_2 \\
 & \leq \alpha' \|a_2\| \|b_1\| \leq 2^n \alpha' \|b_1\|_2 \|a_2\|_2 \leq 2^n \alpha' \|x_1\|_2 \|x_2\|_2,
 \end{aligned}$$

where for the last row we have used the fact that for $a \in C^0$ we have $\|a\| \leq 2^{n/2} \|a\|_2$. Similarly, for the third term of the last line in (4) we have

$$(7) \quad |\tau(u(b_2\xi_2a_1\xi_1))| \leq 2^n \alpha' \|x_1\|_2 \|x_2\|_2.$$

Finally, for the fourth term of the last line in (4), by (3) and the fact that $\xi_1b_1, \xi_2b_2 \in X_0$, we get

$$\begin{aligned}
 (8) \quad & |\tau(u(\xi_1b_2)u(\xi_2b_1))| \leq \alpha' \|\xi_1b_2\|_2 \|\xi_2b_1\|_2 \\
 & \leq \alpha' \|b_1\| \|b_2\| \leq 2^n \alpha' \|b_1\|_2 \|b_2\|_2 \leq 2^n \alpha' \|x_1\|_2 \|x_2\|_2.
 \end{aligned}$$

By combining (4) – (8), we thus obtain for all $\xi_1, \xi_2 \in X$ and $x_1, x_2 \in C$

$$(9) \quad |\tau(x_1\xi_1x_2\xi_2)| \leq 4 \cdot 2^n \alpha' \|x_1\|_2 \|x_2\|_2 \leq \alpha \|x_1\|_2 \|x_2\|_2$$

Similarly, if $\eta \in Y$ and $x = a + bu \in C$, then $\eta b \in Y_0$, $\|\eta b\|_1 \leq 2^n \|b\| \leq 2^n \|x\|$, and by the second part of (1) and (3') we get

$$(10) \quad |\tau(x\eta)| \leq |\tau(a\eta)| + |\tau(u(\eta b))|$$

$$\leq \alpha' \|a\| + \delta \|\eta b\|_1 \leq \alpha' \|x\| + 2^n \delta \|x\| \leq \alpha \|x\|,$$

showing that C satisfies conditions (a) and (b) of the statement.

If we now assume (a) and (b) are satisfied and combine them with the identity

$$\begin{aligned} \|q_j \xi q_i\|_2^2 &= \tau(q_i \xi^* q_j \xi) = \tau((q_i - \tau(q_i)) \xi^* q_j \xi) + \tau(q_i) \tau(q_j \xi \xi^*) \\ &= \tau((q_i - \tau(q_i)) \xi^* (q_j - \tau(q_j)) \xi) + \tau(q_j) \tau((q_i - \tau(q_i)) \xi^* \xi) \\ &\quad + \tau(q_i) \tau((q_j - \tau(q_j)) \xi \xi^*) + \tau(q_i) \tau(q_j) \tau(\xi^* \xi) \end{aligned}$$

then we get:

$$|\|q_j \xi q_i\|_2^2 - \tau(q_i) \tau(q_j) \|\xi\|_2^2| \leq 2^{-n} \alpha + 2 \cdot 2^{-n} \alpha = 3 \cdot 2^{-n} \alpha.$$

This proves that (a) and (b) imply (a'), while (b') is trivial from (b) and (c') from (a'). Finally, (d') follows from the first part of (c'), via the Cauchy-Schwartz inequality:

$$\begin{aligned} \|q_i \xi q_i\|_1 &= \sup\{|\tau(q_i \xi q_i x)| \mid x \in M, \|x\| \leq 1\} \\ &\leq \|q_i \xi q_i\|_2 \sup\{\|q_i x\|_2 \mid x \in M, \|x\| \leq 1\} = \|q_i \xi q_i\|_2 \|q_i\|_2. \end{aligned}$$

□

3.7. Corollary. *Let M be a II_1 von Neumann algebra and $A \subset M$ a MASA in M . Let $X \subset M \ominus A$, $Y \subset M$ be finite sets, $n \geq 1$ an integer and $\alpha > 0$. There exists a partition of 1 with projections $q_1, \dots, q_{2^n} \in A$ of trace 2^{-n} such that if C denotes the algebra generated by $\{q_i\}_i$ then for all $x \in X$, $y \in Y$ and $i = 1, 2, \dots, 2^n$ we have:*

- (a) $|\tau(a_1 x_1 a_2 x_2)| \leq \alpha \|a_1\|_2 \|a_2\|_2$, $\forall x_1, x_2 \in X$, $\forall a_1, a_2 \in C \ominus \mathbb{C}$
- (b) $|\tau(y q_i) - \tau(y) \tau(q_i)| \leq \alpha$, $\forall i$, $\forall y \in Y \cup X X^*$.

Moreover, we have for all $x \in X$ and $1 \leq i, j \leq 2^n$:

- (c) $|\|q_i x q_j\|_2^2 - \|x\|_2^2 \tau(q_i) \tau(q_j)| \leq 3 \cdot 2^{-n} \alpha$.

Proof. Since A is a MASA we have $A' \cap M = A$ and since M is II_1 , A must be diffuse. Thus, Theorem 3.6 applies.

□

3.8. Lemma. *Let M be a finite von Neumann algebra and $A \subset M$ a MASA. Given any separable von Neumann subalgebra $Q_0 \subset M$, there exists a separable von Neumann algebra $Q \subset M$ that contains Q_0 , such that $E_A(Q) = A \cap Q$ (i.e. Q and $A \subset M$ make a commuting square) and $A_0 = A \cap Q$ is maximal abelian in Q .*

Proof. First note that by Theorem 3.6, given any countable set $X = \{x_n\}_n \subset M$ there exists a countably generated abelian von Neumann subalgebra $B_1 \subset M$ such that $E_{B'_1 \cap M}(x_n) \subset B_0$, $\forall n$. Indeed, this is obtained by taking B_1 to be generated by the set $\{E_A(x_n)\}_n$ and partitions $\{p_{j,m}^n\}_j \subset A$, $n, m \geq 1$, satisfying $\|\sum_j p_{j,m}^n(x_n - E_A(x_n))p_{j,m}^n\|_2 \leq 2^{-m}$ (which exist by Theorem 3.6). By taking X to be a $\|\cdot\|_2$ -dense subset in the unit ball of Q_0 , it follows that if we take Q_1 to be the von Neumann algebra generated by B_1 and Q_0 then we have:

(1) $B_1 \subset A$ is separable and satisfies $E_{B'_1 \cap M}(Q_0) \subset B_1$; Q_1 is generated by Q_0, B_1 and is separable;

Using this first part, it follows that we can construct recursively an increasing sequence of inclusions of separable von Neumann algebra $B_n \subset Q_n$, $n \geq 1$, satisfying the properties:

(2) For each $n \geq 1$, $E_{B'_n \cap M}(Q_{n-1}) \subset B_n$ and Q_n is the von Neumann algebra generated by B_n and Q_{n-1} .

If we now define $A_0 = \overline{\cup_n B_n}^w$ and $Q = \overline{\cup_n Q_n}^w$, then all required conditions are clearly satisfied.

□

3.9. Theorem. *Let M_n be a sequence of finite factors with $\dim M_n \rightarrow \infty$ and for each n , let $A_n \subset M_n$ be a MASA. Denote by $\mathbf{A} = \Pi_\omega A_n \subset \Pi_\omega M_n$. Let $Q \subset \Pi_\omega M_n$ be an arbitrary separable von Neumann subalgebra such that $E_{\mathbf{A}}(Q) = \mathbf{A} \cap Q$, i.e. Q and $\mathbf{A} \subset \Pi_\omega M_n$ make a commuting square, and denote $B_1 = \mathbf{A} \cap Q$. There exists a diffuse von Neumann subalgebra $B_0 \subset \mathbf{A}$ such that B_0 is 2-independent to $Q \ominus B_1$, more precisely:*

- (a) $\tau(x_1 a_1 x_2 a_2) = 0$, $\forall x_1 \in Q$, $x_2 \in Q \ominus B_1$, $a_1 \in B_0 \ominus \mathbb{C}$, $a_2 \in B_0$.
- (b) $\tau(xa) = \tau(x)\tau(a)$, $\forall x \in Q$, $a \in B_0$. Equivalently, B_0, B_1 are τ -independent: $(B_0 \vee B_1, \tau) \simeq (B_0, \tau) \overline{\otimes} (B_1, \tau)$.
- (c) $\|exf\|_2^2 = \tau(e)\tau(f)\|x\|_2^2$, $\forall x \in Q \ominus B_1$, $e, f \in \mathcal{P}(B_0)$.

Proof. Using Corollary 3.7 we construct recursively an increasing sequence of partitions of 1 in \mathbf{A} , $\{e_j^n \mid 1 \leq j \leq 2^n\} \subset \mathbf{A}$, of trace $\tau(e_j^n) = 2^{-n}$, $\forall j$, with $e_{2i-1}^{n+1} + e_{2i}^{n+1} = e_i^n$ for all $1 \leq i, j \leq 2^n$, such that the representations $e_j^n = (e_{j,m}^n)_m$ have components $e_{j,m}^n \in \mathcal{P}(A_m)$ satisfying better and better the required conditions (see end of proof of 5.3.1). We leave the details as an exercise.

□

4. ASYMPTOTIC FREENESS AND KADISON-SINGER FOR SINGULAR MASAS

Recall from [D1] that a MASA A in a finite von Neumann algebra M is *singular* if the only unitaries in M that normalize A are the unitaries in A , i.e. $\mathcal{N}_M(A) = \mathcal{U}(A)$. It is easy to see that the existence of such a MASA in a finite von Neumann algebra M implies M is necessarily of type II_1 (unless $M = A$). For concrete examples of singular MASAs in II_1 factors, see [D1], [P2] and Section 5.1 below. Note that by [P3], any separable II_1 factor has singular MASAs. The prototype singular MASA in the hyperfinite II_1 factor R is the abelian von Neumann algebra $L(\mathbb{Z})$ generated by the canonical unitary implementing the Bernoulli action $\mathbb{Z} \curvearrowright X = [0, 1]^{\mathbb{Z}}$, in the representation of R given by the Murray-von Neumann group measure space construction [MvN1], $R = L^\infty(X) \rtimes \mathbb{Z}$.

The main result of this section shows that if $A \subset M$ is a singular MASA in a finite von Neumann algebra, then the associated ultrapower MASA inclusion $A^\omega \subset M^\omega$ satisfies the paving property, and thus the KS property as well. In fact, we prove that given any countable set $F = F^* \subset M^\omega$ perpendicular to A^ω , there exists a diffuse subalgebra B_0 of A^ω which is *free independent* to F in the sense of [V], i.e., any alternating word in $F, B_0 \ominus \mathbb{C}$ has trace 0. Note that this amounts to F, B_0 being n -independent $\forall n$. In particular, due to calculations of norms in [V], this implies that any $x \in F$ which is a selfadjoint element with two point spectrum, has the property that any partition of small mesh with projections in A_0 , provides a paving of x . As we saw in 2.3.2°, this is sufficient to ensure that ANY $x \in M^\omega$ with $E_{A^\omega}(x) = 0$ can be paved with finite partitions in A^ω , and thus $A^\omega \subset M^\omega$ satisfies the Kadison-Singer property.

More precisely, we prove the following:

4.1. Theorem. *Let $\mathcal{S} = \{A_n \subset M_n\}_n$ be a sequence of singular MASAs in finite von Neumann algebras and denote $\mathbf{M} = \mathbf{M}(\mathcal{S}, \omega) = \Pi_\omega M_n$, $\mathbf{A} = \mathbf{A}(\mathcal{S}, \omega) = \Pi_\omega A_n$. Then we have:*

- (a) *If $X \in \mathbf{M} \ominus \mathbf{A}$ is a countable set, then there exists a diffuse von Neumann subalgebra $B_0 \subset \mathbf{A}$ such that X and B_0 are free independent, more precisely: $\tau(x_0 \prod_{i=1}^k y_i x_i) = 0$, for all $k \geq 1$ and all $x_i \in X$, $1 \leq i \leq k-1$, $x_0, x_k \in X \cup \{1\}$, $y_i \in B_0 \ominus \mathbb{C}$, $1 \leq i \leq k$.*
- (b) *Let $B \subset \mathbf{M}$ be a countable generated von Neumann subalgebra such that $E_{\mathbf{A}}(B) = \mathbf{A} \cap B$, i.e. B and \mathbf{A} make a commuting square. Then there exists a diffuse von Neumann subalgebra $B_0 \subset \mathbf{A}$ such that if we denote by $B_1 = \mathbf{A} \cap B$, then $B \vee B_0 = B *_B B_0 \otimes B_1$. In particular, if $B \perp \mathbf{A}$ then $B \vee B_0 = B *_B B_0$.*

One should note that by Lemma 3.8, any separable subalgebra $Q \subset \mathbf{M}$ is contained

in a larger von Neumann subalgebra $B \subset \mathbf{M}$ satisfying the commuting square condition in part (b) of 4.1.

The above theorem shows that any countable set X perpendicular to \mathbf{A} , can be paved with partitions in \mathbf{A} that are free with respect to X . For special type of elements, such as unitaries with 0-expectation on \mathbf{A} , any such “free paving” diminishes the operator norm, due to Kesten-type phenomena [Ke] and Voiculescu’s calculations of spectra for products of free-independent variables [V]:

4.2. Corollary. *Let $A_n \subset M_n$, \mathbf{A} , \mathbf{M} be as above. Then we have:*

1° *If $u \in \mathbf{M}$ is a unitary element such that $E_{\mathbf{A}}(u) = 0$, then for any $n \geq 1$, there exists a partition of 1 with n projections $q_1, \dots, q_n \in \mathbf{A}$ such that $\|\sum_{i=1}^n q_i u q_i\| \leq (\sqrt{n-1} + 1)/n$.*

2° *If e is a projection in \mathbf{M} such that $E_{\mathbf{A}} = \tau(e)1$ and $\tau(e) \leq 1/2$, then for any $n \geq \tau(e)^{-1}$ there exists a partition of 1 with n projections $q_1, \dots, q_n \in \mathbf{A}$ such that $\|\sum_{i=1}^n q_i e q_i - \tau(e)1\| \leq 2/\sqrt{n}$.*

As we saw in Proposition 2.3, the paving of projections that expect on scalars on ultrapowers of MASAs, is in fact sufficient to ensure paving of any element, so from 4.2 above we deduce:

4.3. Corollary (Kadison-Singer for ultraproduct of singular MASAs). *Let $A_n \subset M_n$, \mathbf{A} , \mathbf{M} be as above. Then the inclusion $\mathbf{A} \subset \mathbf{M}$ satisfies the KS property. Thus, any pure state on \mathbf{A} has a unique state extension to \mathbf{M} and $E_{\mathbf{A}}$ is the unique conditional expectation of \mathbf{M} onto \mathbf{A} . Moreover, $\mathbf{A} \subset \mathbf{M}$ has the uniform paving property, with paving size $n(\varepsilon)$ majorized by a scalar multiple of ε^{-6} .*

The proof of Theorem 4.1 will follow quite closely the type of arguments that we have developed in [P8]. We will thus use extensively the notations, terminology and technical lemmas from that paper, which we recall here in details, for the reader’s convenience.

4.4. Notation. Let \mathcal{M} be a von Neumann algebra. If $v \in \mathcal{M}$ is a partial isometry with $v^*v = vv^*$, $X \subset \mathcal{M}$ is a subset and $k \leq n$ are nonnegative integers then denote $X_v^{0,n} \stackrel{\text{def}}{=} X$ and $X_v^{k,n} \stackrel{\text{def}}{=} \{x_0 \prod_{i=1}^k v_i x_i \mid x_i \in X, 1 \leq i \leq k-1, x_0, x_k \in X \cup \{1\}, v_i \in \{v^j \mid 1 \leq |j| \leq k\}\}$.

4.5. Lemma. *Let A be a singular MASA in the finite von Neumann algebra M . Let $\varepsilon > 0$, $n \geq 1$ an integer and $F \subset M$ a finite set such that $E_A(x) = 0$, for all $x \in F = F^*$. There exists a partial isometry $v \in A$ such that $\tau(vv^*) > 1/2$ and $\|E_A(x)\|_1 \leq \varepsilon$, $\forall x \in \bigcup_{k=1}^n F_v^{k,n}$.*

Proof. It is clearly sufficient to prove the statement in case $\|x\| \leq 1$, $\forall x \in F$. Let $\delta > 0$. Denote $\varepsilon_0 = \delta$, $\varepsilon_k = 2^k \varepsilon_{k-1}$, $k \geq 1$. Let $\mathcal{W} = \{v \in A \mid vv^* \in \mathcal{P}(A), \|E_A(x)\|_1 \leq \varepsilon_k \tau(v^*v), \forall 1 \leq k \leq n, \forall x \in F_v^{k,n}\}$. Endow \mathcal{W} with the order \leq in which $w_1 \leq w_2$ iff $w_1 = w_2 w_1^* w_1$. (\mathcal{W}, \leq) is then clearly inductively ordered. Let v be a maximal element in \mathcal{W} . Assume $\tau(v^*v) \leq 1/2$ and denote $p = 1 - v^*v$. If w is a partial isometry in Ap and $u = v + w$, then for $x = x_0 \prod_{i=1}^k u_i x_i \in F_u^{k,n}$ we have

$$(1) \quad x = x_0 \prod_{i=1}^k v_i x_i + \sum_{\ell} \sum_i z_{0,i} \prod_{j=1}^{\ell} w_{i_j} z_{j,i},$$

where the sum is taken over all $\ell = 1, 2, \dots, k$ and all $i = (i_1, \dots, i_{\ell})$, with $1 \leq i_1 < \dots < i_{\ell} \leq k$, and where $w_{i_j} = w^s$ whenever $v_j = v^s$, $z_{0,i} = x_0 v_1 x_1 \cdots x_{i_1-1} p$, $z_{j,i} = p x_{i_j} v_{i_j+1} \cdots v_{i_{j+1}} x_{i_{j+1}} p$, for $1 \leq j < \ell$, and $z_{\ell,i} = p x_{i_{\ell}} v_{i_{\ell}+1} \cdots v_k x_k$.

By Corollary 3.3, given any $\alpha > 0$ there exists a projection q in Ap such that

$$(2) \quad \|qzq - E_{Ap}(pzp)q\|_{1,pMp} < \alpha \tau_{pMp}(q),$$

$$\|E_{Ap}(pzp)q\|_{1,pMp} \leq (1 + \alpha) \|E_{Ap}(z)\|_{1,pMp} \tau(q)$$

for all z of the form $z_{j,i}$, for some $i = (i_1, \dots, i_{\ell})$, $1 \leq j \leq \ell - 1$, $\ell \geq 2$.

Since for $y_1, y_2, y \in M$ with $\|y_1\| \leq 1, \|y_2\| \leq 1$ we have $\|E_A(y_1 y y_2)\|_1 \leq \|y_1 y y_2\|_1 \leq \|y\|_1$, it follows that for any $\ell \geq 2$ we have:

$$(3) \quad \|E_A(z_{0,i} w_{i_1} z_{1,i} w_{i_2} z_{2,i} \cdots w_{i_{\ell}} z_{\ell,i})\|_1 \\ \leq \|w_{i_1} z_{1,i} w_{i_2}\|_1 = \|q z_{1,i} q\|_1 = \|q z_{1,i} q\|_{1,pMp} \tau(p),$$

which by applying consecutively to $z = z_{1,i}$ the two inequalities in (2), is further majorised by

$$(4) \quad \leq (\|E_{Ap}(z_{1,i})q\|_{1,pMp} + \alpha \tau_{pMp}(q)) \tau(p) \\ \leq (1 + \alpha) \|E_{Ap}(z_{1,i})\|_{1,pMp} \tau_{pMp}(q) \tau(p) + \alpha \tau_{pMp}(q) \tau(p) \\ = (1 + \alpha) (\|E_A(z_{1,i})\|_1 \tau(p)^{-1}) (\tau(q) \tau(p)^{-1}) \tau(p) + \alpha \tau(q) \\ = (1 + \alpha) \|E_A(p x_{i_1} v_{i_1+1} \cdots v_{i_2} x_{i_2} p)\|_1 \tau(p)^{-1} \tau(q) + \alpha \tau(q).$$

But since p lies in A , we have $\|E_A(pyp)\|_1 = \|p E_A(y) p\|_1 \leq \|E_A(y)\|_1$ for any $y \in M$. Also, since $1 \leq i_1 < i_2 \leq \ell$ and $i_2 - i_1 \leq k - 1$, the element $y = x_{i_1} v_{i_1+1} \cdots v_{i_2} x_{i_2}$

belongs to $F_v^{m,n}$ with $m = i_2 - i_1 - 1 \leq k - 2$. Altogether, it follows that the last term in (4) is majorized by

$$\begin{aligned}
 (5) \quad & (1 + \alpha) \|E_A(x_{i_1} v_{i_1+1} \cdots v_{i_2} x_{i_2})\|_1 \tau(p)^{-1} \tau(q) + \alpha \tau(q) \\
 & \leq (1 + \alpha) \varepsilon_{k-2} \tau(vv^*) \tau(p)^{-1} \tau(q) + \alpha \tau(q) \\
 & \leq (1 + \alpha) \varepsilon_{k-2} \tau(q) + \alpha \tau(q),
 \end{aligned}$$

where the last inequality is due to the fact that $\tau(vv^*) \leq 1/2$ implies $\tau(vv^*)/\tau(p) \leq 1$. If we now take $\alpha \leq \varepsilon_0/4$, from the first term of (3) and last term of (5), we get that for all $i = (i_1, \dots, i_\ell)$ with $\ell \geq 2$ we have

$$(6) \quad \|E_A(z_{0,i} w_{i_1} z_{1,i} w_{i_2} z_{2,i} \dots w_{i_\ell} z_{\ell,i})\|_1 \leq 2\varepsilon_{k-2} \tau(q).$$

Since $2\varepsilon_{k-2} \leq \varepsilon_{k-1}$ and since there are at most $\sum_{i=2}^k \binom{k}{i} = 2^k - k - 1$ elements in the sum in (1) for which $\ell \geq 2$, from (6) we get

$$(7) \quad \sum_{\ell \geq 2} \sum_i \left\| E_A \left(z_{0,i} \prod_{j=1}^{\ell} w_{i_j} z_{j,i} \right) \right\|_1 \leq (2^k - k - 1) \varepsilon_{k-1} \tau(q)$$

Finally, from the sum on the right hand side of (1) we will now estimate the terms with $\ell = 1$. These are terms which are obtained from $x_0 v_1 x_1 v_2 x_2 \dots v_k x_k$ by replacing exactly one v_i by w_i , so they are of the form $z = z_{0,i} w_i z_{1,i}$, where $i = 1, 2, \dots, k$, $z_{0,i} = x_0 v_1 x_1 \dots v_{i-1} x_{i-1} p$, $z_{1,i} = p x_i v_{i+1} \dots v_k x_k$ and $w_i = w^s$ if $v_i = v^s$. Note that there are k of them.

One should notice at this point that in the above estimates we only used the fact that $w^* w = w w^* = q$ and that A is a MASA, not the actual form of w , nor the fact that A is singular. We will make the appropriate choice for w now, to get the necessary estimates for these last terms. The singularity assumption on A will play a crucial role, due to the following:

4.6. Lemma. *Let $A \subset M$ be a singular MASA. Let $Y_1 = Y_1^* \subset M \ominus A$, $Y_2 \subset M$ be finite sets and $q \in A$ a nonzero projection. Given any $\beta > 0$ and $n \geq 1$ there exists a unitary element $w \in Aq$ such that $\|E_A(y_1 w^i y_2)\|_1 < \beta$ and $\|E_A(y_2 w^i y_1)\|_1 < \beta$, for all $y_1 \in Y_1, y_2 \in Y_2$, $0 < |i| \leq n$.*

Proof. We may clearly assume $\|y_i\| \leq 1$, $\forall y_i \in Y_i$, $i = 1, 2$. Let $\langle M, e_A \rangle$ be the Jones basic construction von Neumann algebra of the inclusion $A \subseteq^{e_A} M$, endowed with its canonical (semifinite) trace $Tr_{\langle M, e_A \rangle}$. Consider the semifinite von Neumann algebra $\mathcal{M} = \langle M, e_A \rangle^{2n}$ and denote by Tr the trace on \mathcal{M} defined by $Tr(x_1, x_2, \dots, x_{2n}) = \sum_j Tr_{\langle M, e_A \rangle}(x_j)$. Let $K_0 \subset \mathcal{M}$ denote the convex hull of the set $\{(w^j(\sum_{y_1 \in Y_1} y_1^* e_A y_1)w^{-j})_{1 \leq |j| \leq n} \mid w \in \mathcal{U}(Aq)\} \subset (q\langle M, e_A \rangle q)^{2n} \subset \mathcal{M}$. One should notice right away that each j 'th component z_j , $1 \leq |j| \leq n$, of an element $z = (z_j)_j \in K_0$ satisfies $z_j = qz_jq$ and $e_A z_j = 0 = z_j e_A$ (the latter because $e_A w^j y_1 e_A = w^j e_A y_1 e_A = w^j E_A(y_1) e_A = 0$, $\forall y_1 \in Y_1$; similarly $e_A y_1 w^j e_A = 0$).

Note further that K_0 is bounded both in the operator norm on \mathcal{M} (by $\sum_{y_1} \|y_1\|^2 \leq |Y_1|$) and in the Hilbert-norm $\|\cdot\|_{2, Tr}$ on \mathcal{M} (by $\sum_{y_1} \|E_A(y_1^* y_1)\|_2^2 \leq |Y_1|$). Thus, its weak closure $K = \overline{K_0}^w$ is a weakly compact bounded subset in both \mathcal{M} and $L^2(\mathcal{M}, Tr)$. In particular, K contains a unique element $b \in K$ with $\|b\|_{2, Tr} = \min\{\|z\|_{2, Tr} \mid z \in K\}$.

Note also that the group $\mathcal{U}(Aq)$ acts on K_0 by $\sigma_w((x_j)_{1 \leq |j| \leq n}) = (w^j x_j w^{-j})_j$, $\forall w \in \mathcal{U}(Aq)$, and that this action preserves the Hilbert norm $\|\cdot\|_{2, Tr}$. Thus, σ extends to an action of $\mathcal{U}(Aq)$ on K , still denoted by σ . Since $\|\sigma_w(b)\|_{2, Tr} = \|b\|_{2, Tr}$, by the uniqueness of b as the element of minimal norm in K , it follows that $\sigma_w(b) = b$, $\forall w \in \mathcal{U}(Aq)$. Thus, if $b = (b_j)_j$ are the $2n$ components of b , then for each j with $1 \leq |j| \leq n$, we have $b_j w^j = w^j b_j$, for all $w \in \mathcal{U}(Aq)$. Since any unitary element in Aq can be expressed as a j 'th power of a unitary in Aq , it follows that $u b_j = b_j u$, $\forall u \in \mathcal{U}(Aq)$. But since any element in Aq is a linear combination of unitary elements in Aq , this implies $b_j \in Aq' \cap q\langle M, e_A \rangle q = A' \cap q\langle M, e_A \rangle q$. But by (1.4 in [P12]), the supremum of finite projections in $A' \cap q\langle M, e_A \rangle q$ is equal to the supremum of the projections $q v e_A v^* q$ with $v \in \mathcal{N}(A)$. Since A is singular in M , this implies $b_j = e_A b_j e_A$. But $e_A b_j = 0$, and so $b_j = 0$, $\forall j$.

We have thus proved that $0_{\mathcal{M}} = (0, \dots, 0) \in K$. This implies that for any $\beta > 0$ there exists $w \in \mathcal{U}(Aq)$ such that

$$Tr(w^j(y_1 e_A y_1^*) w^{-j}(\sum_{y_2 \in Y_2} y_2 e_A y_2^*)) < \beta^2,$$

for all $y_1 \in Y_1$, where the sum is taken over $y_2 \in Y_2$. Indeed, for if not then

$$Tr(w^j(\sum_{y_1 \in Y_1} y_1 e_A y_1^*) w^{-j}(\sum_{y_2 \in Y_2} y_2 e_A y_2^*)) \geq \beta^2,$$

for all $w \in \mathcal{U}(Aq)$. By taking convex combinations over $w \in \mathcal{U}(Aq)$ and then weak limits, this implies $Tr(b \sum_{y_2 \in Y_2} y_2 e_A y_2^*) \geq \beta^2$, $\forall b \in K$, in particular for $b = 0$, thus $0 \geq \beta^2 > 0$, a contradiction.

In particular, such a $w \in \mathcal{U}(Aq)$ will satisfy $Tr(w^j(y_1 e_A y_1^*) w^{-j}(\sum_{y_2 \in Y_2} y_2 e_A y_2^*)) < \beta^2$, for all $y_1 \in Y_1, y_2 \in Y_2$ and all j with $1 \leq |j| \leq n$. By taking into account the

definitions of $\|\cdot\|_1$ and of $Tr_{\langle M, e_A \rangle}$, and by using the Cauchy-Schwartz inequality in $(\langle M, e_A \rangle, Tr)$, we thus get the estimates

$$\begin{aligned} \|E_A(y_1 w^j y_2)\|_1 &= \sup\{|\tau(y_1 w^j y_2 a)| \mid a \in A, \|a\| \leq 1\} \\ &= \sup\{|Tr(e_A y_1 w^j y_2 e_A a)| \mid a \in A, \|a\| \leq 1\} \\ &\leq Tr(e_A y_2^* w^{-j} y_1^* e_A y_1 w^j y_2 e_A)^{1/2} Tr(e_A)^{1/2} \\ &= Tr(w^{-j} y_1^* e_A y_1 w^j y_2 e_A y_2^*)^{1/2} \leq \beta, \end{aligned}$$

and similarly $\|E_A(y_2 w^j y_1)\|_1 \leq \beta$, for all $y_1 \in Y_1 = Y_1^*$, $y_2 \in Y_2$, $j = \pm 1, \pm 2, \dots, \pm n$. \square

End of proof of 4.5: Denote by Z the set of elements of the form $x_0 v_1 x_1 \dots v_{i-1} x_{i-1} p$, or $p x_i v_{i+1} \dots v_k x_k$, for all possible choices arising from elements in $\bigcup_{k=1}^n F_v^{k,n}$. By applying Lemma 4.6 to $\beta = \varepsilon_{k-1} \tau(q)/2k$, $n \geq 1$ and $Y_2 = Z \cup Z^* \cup \{E_A(z) \mid z \in Z \cup Z^*\}$, $Y_1 = \{y_2 - E_A(y_2) \mid y_2 \in Y_2\}$, it follows that there exists $w \in \mathcal{U}(Aq)$ such that

$$\begin{aligned} (8) \quad \|E_A((x_0 v_1 x_1 \dots v_{j-1} x_{j-1} - E_A(x_0 v_1 x_1 \dots v_{j-1} x_{j-1} p) w_j x_j v_{j+1} \dots v_k x_k)\|_1 \\ \leq \varepsilon_{k-1} \tau(q)/2k, \end{aligned}$$

$$\begin{aligned} (8') \quad \|E_A(x_0 v_1 x_1 \dots v_{j-1} x_{j-1} w_j (x_j v_{j+1} \dots v_k x_k - E_A(p x_j v_{j+1} \dots v_k x_k)))\|_1 \\ \leq \varepsilon_{k-1} \tau(q)/2k. \end{aligned}$$

Thus, for each element with $\ell = 1$ in the summation $\sum_\ell \sum_i z_{0,i} \prod_{j=1}^\ell w_{i_j} z_{j,i}$ in (1), i.e., of the form $x_0 v_1 x_1 \dots v_{j-1} x_{j-1} w_j x_j v_{j+1} \dots v_k x_k$, we have the estimate:

$$\begin{aligned} (9) \quad \|E_A(x_0 v_1 x_1 \dots v_{j-1} x_{j-1} w_j x_j v_{j+1} \dots v_k x_k)\|_1 \\ \leq 2\varepsilon_{k-1} \tau(q)/2k + \|E_A(x_0 v_1 x_1 \dots v_{j-1} x_{j-1}) w_j E_A(x_j v_{j+1} \dots v_k x_k)\|_1 \\ \leq \varepsilon_{k-1} \tau(q)/k + \gamma, \end{aligned}$$

where γ is the minimum of $\|E_A(x_0 v_1 x_1 \dots v_{j-1} x_{j-1}) q\|_1$, $\|q E_A(x_j v_{j+1} \dots v_k x_k)\|_1$, which by the second inequality in (2) is majorized by the minimum between $\|(1 + \alpha) E_A(p x_0 v_1 x_1 \dots v_{j-1} x_{j-1} p)\|_1 \tau(q)$ and $(1 + \alpha) \|E_A(p x_j v_{j+1} \dots v_k x_k p)\|_1 \tau(q)$. Since

$\|E_A(pyp)\|_1 = \|pE_A(y)p\|_1 \leq \|E_A(y)\|_1$, the latter is majorized by the minimum between $(1+\alpha)\|E_A(x_0v_1x_1\ldots v_{j-1}x_{j-1})\|_1\tau(q)$, $(1+\alpha)\|E_A(x_jv_{j+1}\ldots v_kx_k)\|_1\tau(q)$. Both elements $x_0v_1x_1\ldots v_{j-1}x_{j-1}$, $x_jv_{j+1}\ldots v_kx_k$ belong to some $F_v^{j,n}$ with $j \leq k-1$, and at least one of them with $j \neq 0$. Thus, by the properties of v we have $\gamma \leq (1+\alpha)\varepsilon_{k-1}\tau(vv^*)\tau(q)$. Since α was taken $\leq \varepsilon_0/4 \leq 1/4$, one gets $\gamma \leq \varepsilon_{k-1}$.

Hence, the last term in (9) is majorized by $\varepsilon_{k-1}\tau(q)/k + \varepsilon_{k-1}$. Since there are k terms with $\ell = 1$, obtained by taking $j = 1, \dots, k$, by summing up over j in (9) and combining with (7), we deduce by applying E_A to (1) the following final estimate:

$$(10) \quad \|E_A(x)\|_1 \leq \|E_A(x_0\Pi_{i=1}^k v_i x_i)\|_1 + \Sigma_\ell \Sigma_i \|E_A(z_{0,i}\Pi_{j=1}^\ell w_{i,j} z_{j,i})\|_1$$

$$\leq \varepsilon_k \tau(vv^*) + (2^k - k - 1)\varepsilon_{k-1}\tau(q) + (k+1)\varepsilon_{k-1}\tau(q)$$

$$= \varepsilon_k \tau(vv^*) + \varepsilon_k \tau(w w^*) = \varepsilon_k \tau((v+w)(v+w)^*).$$

But this contradicts the maximality of $v \in \mathcal{W}$.

We conclude that $\tau(v^*v) > 1/2$. If we now take $\delta \leq \varepsilon/2^{n^2}$, then $\varepsilon_n = 2^{1+2+\dots+n}\delta < 2^{n^2}\delta \leq \varepsilon$ and the statement follows. \square

Lemma 4.7. *Let $A_n \subset M_n$, $\mathbf{A} \subset \mathbf{M}$ be as in 4.1. Let $X \subset \mathbf{M} \ominus \mathbf{A}$ be a countable set. Then there exists a partial isometry w in \mathbf{A} such that $\tau(w w^*) \geq 1/2$ and $E_{\mathbf{A}}(x) = 0$, $\forall n \geq k \geq 1$, $\forall x \in X_w^{k,n}$.*

Proof. Let $X = \{x^k\}_k$ and let $x^k = (x_n^k)_n$ be a representation of $x^k \in \Pi_\omega M_n$, which we can take so that $x_n^k \in M_n$ satisfy $E_{A_n}(x_n^k) = 0$, for all k . By applying Lemma 4.4 for the inclusion $A_n \subset M_n$, the positive element $\varepsilon = 2^{-n}$, the integer n and the finite set $X_n = \{x_n^k \mid k \leq n\}$, we get a partial isometry w_n in A_n such that $\tau(w_n^* w_n) \geq 1/2$ and

$$\|E_{A_n}(x)\|_1 \leq 2^{-n}, \forall x \in \bigcup_{k \leq n} (X_n)_{w_n}^{k,n}.$$

But then $w = (w_n)$ clearly satisfies the required conditions. \square

Proof of 4.1. Since 4.1(b) is an immediate consequence of part 4.1(a), we only need to prove the latter. To do this, we construct recursively a sequence of partial isometries $v_1, v_2, \dots \in \mathbf{A}$ such that

- (i) $v_{j+1}v_j^*v_j = v_j$ and $\tau(v_jv_j^*) \geq 1 - 1/2^j$, $\forall j \geq 1$.
- (ii) $E_{\mathbf{A}}(x) = 0$, $\forall n \geq k \geq 1$, $\forall x \in X_{v_j}^{k,n}$

Assume we have constructed v_j for $j = 1, \dots, m$. If v_m is a unitary element, then we let $v_j = v_m$ for all $j \geq m$. If v_m is not a unitary element, then let $f = 1 - v_m^* v_m \in \mathbf{A}$. Note that $E_{\mathbf{A}}(x') = 0$, for all $x' \in X' \stackrel{\text{def}}{=} \bigcup_{k \leq n} f X_{v_m}^{k,n} f$. Indeed, because for $a \in \mathbf{A}$ and $x \in X_{v_m}^{k,n}$, we have

$$\tau(f x f a) = \tau(f x a) = \tau((1 - v^* v) x a) = \tau(x a) - \tau(v x v^* a) = 0.$$

This is due to the fact that either $x \in X_{v_m}^{k,n}$ begins or ends with a nonzero power of v_m (when $x = x_0 v_m^{i_1} x_1 \dots v_m^{i_k} x_k$ with either x_0 or x_k equal to 1) or $v x v^* \in X_{v_m}^{k+2, n+2}$, implying that both $\tau(x a) = 0$ and $\tau(v_m x v_m^* a) = 0$ (because v_m satisfies (ii) above).

Thus, if we let $f = (f_n)_n$ with $f_n \in \mathcal{P}(A_n)$ and apply Lemma 4.6 to $A_n f_n \subset f_n M_n f_n$ (which are obviously singular MASAs) and to the countable set $X' \subset \Pi_{\omega} f_n M_n f_n = f \mathbf{M} f$, then we get a partial isometry $u \in \Pi_{\omega} A_n f_n = \mathbf{A} f$ such that $\tau(u u^*) \geq \tau(f)/2$ and $E_{\mathbf{A} f}(x) = 0$ for all $x \in \bigcup_{k \leq n} (X')_u^{k,n}$. But then $v_{m+1} = v_m + u$ will satisfy both (i) and (ii) for $j = m + 1$.

It follows now from (i) that the sequence v_j converges in the norm $\|\cdot\|_2$ to a unitary element $v \in \mathbf{A}$, which due to (ii) will satisfy the conditions required in part (a) of 4.1. □

To deduce 4.2.1° from 4.1, we'll need the following:

Lemma 4.8. *Let u, v be unitary elements in a finite von Neumann algebra M such that $\tau(u) = 0$ and $\tau(v^j) = 0$, for any non-zero integer j with $|j| \leq n - 1$, for some $n \geq 2$. Assume $\tau(x_0 y_1 x_1 y_2 \dots y_k x_k) = 0$, for any $k \geq 1$ and any choice of $y_i \in \{u, u^*\}$, $x_1, \dots, x_{k-1} \in \{v^j \mid 1 \leq |j| \leq k - 1\}$ and $x_0, x_k \in \{v^j \mid 1 \leq |j| \leq k - 1\} \cup \{1\}$. Then we have:*

- (a) $\{u^* v^j u v^{-j} \mid j = 1, 2, \dots, n - 1\}$ are freely independent Haar unitaries in M .
- (b) $\|\sum_{j=0}^{n-1} v^j u v^{-j}\| \leq \sqrt{n-1} + 1$.

Proof. Part (a) is easy to check and we leave it as an exercise (see e.g. [AO] for similar calculations).

To deduce part (b), recall that by a well known result of Kesten ([Ke]), if w_1, \dots, w_m are freely independent Haar unitaries in a finite von Neumann algebra M , then $\|\sum_{i=1}^m w_i\| = \sqrt{m}$. Thus, by (a) we get:

$$\begin{aligned} \|\sum_{j=0}^{n-1} v^j u v^{-j}\| &= \|u(1 + \sum_{j=1}^{n-1} u^* v^j u v^{-j})\| \\ &= \|1 + \sum_{j=1}^{n-1} v^* v^j u v^{-j}\| \leq 1 + \|\sum_{j=1}^{n-1} u^* v^j u v^{-j}\| = 1 + \sqrt{n-1}. \end{aligned}$$

□

Proof of 4.2. By Theorem 4.1 there exists a diffuse von Neumann subalgebra $B_0 \subset A = A^\omega$ such that any word with alternating letters from $\{u, u^*\}$, $B_0 \ominus \mathbb{C}1$, has trace 0. Let $v \in B_0$ be a unitary element such that $\tau(v^j) = 0$ for $j = 1, 2, \dots, n-1$ and $v^n = 1$. Thus, if $\lambda \in \mathbb{C}$ is a primitive n 'th root of 1 then $v = \sum_{k=0}^{n-1} \lambda^k e_{k+1}$, where $e_k \in B_0$ are spectral projections of v with $\tau(e_k) = 1/n$, $\forall k$. An easy calculation shows that $n^{-1} \sum_{j=0}^{n-1} v^j u v^{-j} = \sum_{k=1}^n e_k u e_k$. But then 4.2.1° follows from 4.8 (b).

To prove 4.2.2° let $B_0 \subset A^\omega$ be free with respect to the von Neumann algebra $\mathbb{C}e + \mathbb{C}(1-e)$. Then the calculation of norms in Section 4 of [V] shows that if q is any projection of trace $1/n$ in B_0 with $1/n \leq \tau(e)$, then $\|qeq - \tau(e)q\| \leq 2/\sqrt{n}$.

□

Proof of 4.3. By Proposition 2.2, in order to prove Corollary 4.3, it is sufficient to prove that any projection $e \in \mathbf{M}$ whose expectation on \mathbf{A} is a scalar multiple of some projection $f \in \mathbf{A}$, can be paved. But this is indeed the case, because $\mathbf{A}f \subset f\mathbf{M}$ is itself an ultraproduct of singular inclusions, for which 4.2 applies.

□

5. FINAL REMARKS

5.1. Examples of singular MASAs. Dixmier's first examples of singular MASAs A in II_1 factors M ([D1]), were constructed from group-subgroup situations, $H \subset G$, as $A = L(H) \subset L(G) = M$, with G infinite conjugacy class (ICC) and $H \subset G$ an abelian subgroup satisfying certain conditions. These conditions are met for instance by wreath product inclusions groups $H \subset G = K \wr H$, with H infinite abelian and K non-trivial and by the inclusions $L(\mathbb{Z}) \subset L(\mathbb{Z} * \Gamma_0)$, for any non-trivial group Γ_0 . Another criterion for singularity of MASAs in factors was found in [P2]. It can be used to recover the previous examples, as well as others. It shows for instance that $A = L^\infty([0, 1])$ is singular in $A * N$ for any finite von Neumann algebra N . It also shows that the group algebra $A = L(H)$ is singular in any crossed product II_1 factor $M = B^{\otimes H} \rtimes H$, arising from a Bernoulli action $H \curvearrowright B^{\otimes H}$, for any non-trivial finite von Neumann “base”-algebra B . In fact, by (3.1 in [P14]), all these MASAs A are singular in the following stronger sense: If $u \in \mathcal{U}(M)$ is so that $uAu^* \cap A$ is diffuse, then $u \in A$. This *absorption* phenomenon from ([P14]) is actually valid for any inclusion $L(H) \subset M = N \rtimes H$, arising from a mixing action of H on a finite von Neumann algebra N .

Another strengthening of the notion of singularity for a MASA $A \subset M$ was emphasized in [P3] and it requires that the only automorphisms of M that normalize A are the inner automorphisms $\text{Ad}(u)$ with $u \in \mathcal{U}(A)$. Such MASAs were called *ultrasingular* in [P3], but we will call them *supersingular* from now on, because they have the property that any two automorphisms of M that coincide on A must differ

by some $\text{Ad}(u)$, with $u \in \mathcal{U}(A)$. Equivalently, embeddings with same range of M into another algebra are uniquely determined by their values on A . It was shown in [P3] that any II_1 factor M whose outer automorphism group $\text{Out}(M)$ is countable (e.g. if M has property T, by [C2]), do have supersingular MASAs.

We note here that results from (Section 4 and 5 of [P15]) show in particular that if one reduces the singular MASA, $A = L(\mathbb{Z}) \subset L([0, 1]^{\mathbb{Z}}) \rtimes \mathbb{Z} = R$, by a projection $p \in A$ which is not fixed by any rotation by a character $\gamma \in \hat{\mathbb{Z}}$, then $Ap \subset pRp \simeq R$ is supersingular. Moreover, if $p, q \in A$ are not conjugate by a rotation then Ap, Aq are distinct singular MASAs in R . More precisely, we have:

5.1.1. Theorem [P15]. *Let H be a torsion free abelian group (such as $H = \mathbb{Z}$) and $H \curvearrowright X = X_0^H$ a Bernoulli H -action. Let $R = L^\infty(X) \rtimes H$ and denote $A = L(H)$. If $p, q \in A$ are non-zero projections and $\theta : pRp \simeq qRq$ is an isomorphism carrying Ap onto Aq , then there exists a character $\gamma \in \hat{H}$ such that θ is the restriction of $\theta_\gamma \in \text{Aut}(R)$ to pRp . Moreover, the only automorphisms of $pRp \simeq R$ that normalize Ap are the restrictions of the automorphisms θ_γ that satisfy $\gamma(Y) = Y$ (a.e.), where $Y \subset \mathbb{T}$ is the subset with characteristic function $\chi_Y = p$. In particular, if $\{p_t \mid t \in (0, 1]\}$ is a family of projections in $L(H)$ with $\tau(p_t) = t$, then $Ap_t \subset p_tRp_t \simeq R$ provide a family of distinct singular MASAs in the hyperfinite II_1 factor, which are supersingular for $t \notin \mathbb{Q}$.*

5.2. Characterizations of singularity for MASAs. Another strengthening of singularity for MASAs was discovered in ([P4]), where it is shown that if A is a diffuse abelian von Neumann algebra and N is any finite von Neumann algebra, then A is maximal amenable (equivalently, maximal injective) in $A * N$. We notice in 5.2.1 below that Theorem 4.1 shows in particular that for any singular MASA $A \subset M$, the ultrapower A^ω is maximal amenable in M^ω , i.e., if $A^\omega \subset P \subset M^\omega$ for some amenable von Neumann algebra P , then $P = A^\omega$. We also provide an alternative characterization of singularity for MASAs in terms of moments, as those MASAs that contain Haar unitaries which are asymptotically free with respect to sets perpendicular to it. For this to happen, asymptotic 4-independence is in fact sufficient. This should be compared to Theorem 3.9 where it was shown that asymptotic 2-independence occurs for any MASA, and to 5.3.1 below, which shows that in fact in arbitrary MASAs asymptotic 3-independence occurs as well.

5.2.1. Theorem. *Let A be a MASA in a finite von Neumann algebra M . The following are equivalent:*

- 1° A is singular in M ;
- 2° A^ω is singular in M^ω ;

3° A^ω is maximal amenable in M^ω ;

4° Given any countable set $X \subset M^\omega \ominus A^\omega$, there exists $B_0 \subset A^\omega$ diffuse such that B_0, X are free independent.

5° Given any finite set $X \subset M \ominus A$, there exists $B_0 \subset A^\omega$ diffuse such that B_0, X are 4-independent.

6° Given any selfadjoint element $x \in M \ominus A$, there exists $B_0 \subset A^\omega$ diffuse such that $B_0, \{x\}$ are 4-independent.

Proof If $u \in \mathcal{N}_M(A)$, then u normalizes A^ω as well, acting non-trivially if $u \notin A$. Thus, A^ω singular implies A singular. The converse is implicit in [P13] (due to Remark 5.2 in [P3]). Indeed, for if $u \in \mathcal{N}(A^\omega)$ is not in A^ω , then there exists a non-zero projection $q \in A^\omega$ such that $uqu^*q = 0$ and u, q can be represented by sequences $u = (u_n)_n, q = (q_n)_n$, with u_n unitaries in M , q_n projections in A , such that $u_n q_n u_n^* q_n = 0$. But by ([P13]), by the singularity of $A \subset M$, for each n there exists a unitary element $v_n \in q_n M q_n$ such that $\|E_A(u_n v_n u_n^*)\|_2 \leq \|q_n\|_2/n$. Thus, $v = (v_n)_n \in A^\omega$ satisfies $uvu^* \perp A^\omega$, a contradiction. Thus 1°, 2° are equivalent.

The implication 1° \implies 4° is shown in Theorem 4.1.(b), and 4° \implies 5° \implies 6° are trivial. To see that 5° \implies 1°, assume A is not singular and let $v \in M$ be a partial isometry such that vv^*, v^*v are mutually orthogonal projections in A and $vAv^* = Avv^*$. If $u \in A^\omega$ would be a Haar unitary that's 4-independent with respect to $X = \{v, v^*\}$, then the equality $vvu^*u^* = u^*vvv^*$ (due to abelianess of A^ω) implies $0 \neq \tau(vv^*) = \tau(vu^*v^*uvuv^*u^*) = 0$, a contradiction. Taking $X = \{v + v^*\}$, this actually proves 6° \implies 1° as well.

The implication 3° \implies 2° is trivial. If N is any von Neumann algebra that strictly contains A^ω , then there exists two orthogonal projections $p_1, p_2 \in A^\omega$ that are equivalent via some partial isometry v in N (exercise!). If $q = 2^{-1}(p_1 + p_2 + v + v^*)$, then q is a projection in N such that $E_{A^\omega}(q) = 2^{-1}p$, where $p = p_1 + p_2$. By Theorem 4.1, there exists a diffuse von Neumann subalgebra $B_0 \subset A^\omega$ such that any alternating word in $B_0 \ominus \mathbb{C}$ and $q - 2^{-1}p$ has trace 0. Thus, the algebras $B_0 p$ and $\mathbb{C}q + \mathbb{C}p$ are free independent, in fact if $u \in B_0 p$ is a Haar unitary then u and $(v + v^*)u(v + v^*)$ generate a copy of the free group factor $L(\mathbb{F}_2)$. Thus, N cannot be amenable. \square

5.3. Controlling moments through incremental patching. In Theorem 3.9, we have proved that if $A \subset M$ is an arbitrary MASA, then for any countable $X \subset M^\omega \ominus A^\omega$, there exists a diffuse von Neumann subalgebra $B_0 \subset A^\omega$ such that B_0 is 2-independent with respect to X . We chose to prove this through a “global” construction of finite dimensional approximations of such a 2-independent B_0 . But we can also prove this result differently, through the method used in the proofs of

the previous section, and which consists in controlling the moments incrementally, by patching “infinitesimal pieces” of an appropriate Haar unitary. This method does use a technical result from Section 3, namely property (d') of Theorem 3.6, but which was already known since [P1] (see also A.1 in [P7]): If M is a finite von Neumann algebra, $A \subset M$ a MASA and $X \subset M \ominus A$ a finite set, then given any $\varepsilon > 0$, there exists a non-zero projection $q \in A$ such that $\|qxq\|_1 \leq \varepsilon\tau(q)$, $\forall x \in X$.

In fact, as shown in 5.3.1 below, the “incremental patching” method can be used to obtain a slightly stronger result for arbitrary MASAs $A \subset M$, showing that one can construct separable, diffuse von Neumann subalgebras $B_0 \subset A^\omega$ that are 3-independent with respect to any given countable set $X \subset M^\omega \ominus A^\omega$. As we saw in Theorem 5.2.1, this is the best one can do for an arbitrary MASA, as existence of a B_0 that’s 4-independent with respect to any given countable set $X \subset M \ominus A$ forces A to be singular (in which case B_0 can even be chosen free independent with respect to the given X).

5.3.1. Theorem. *Let M_n be a sequence of finite factors with $\dim M_n \rightarrow \infty$ and for each n , let $A_n \subset M_n$ be a MASA. Denote by $\mathbf{A} = \Pi_\omega A_n \subset \Pi_\omega M_n$. Let $Q \subset \Pi_\omega M_n$ be an arbitrary separable von Neumann subalgebra such that $E_{\mathbf{A}}(Q) = \mathbf{A} \cap Q$, i.e. Q and $\mathbf{A} \subset \Pi_\omega M_n$ make a commuting square, and denote $B_1 = \mathbf{A} \cap Q$. There exists a diffuse von Neumann subalgebra $B_0 \subset \mathbf{A}$ such that B_0 is 3-independent to $Q \ominus B_1$, more precisley: $\tau(xa) = 0$, $\forall x \in Q, a \in B_0 \ominus \mathbb{C}$; $\tau(x_1 a_1 x_2 a_2) = 0$, $\tau(x_1 a_1 x_2 a_2 x_3 a_3) = 0$, $\forall x_i \in Q \ominus B_1, a_i \in B_0 \ominus \mathbb{C}$ (N.B.: the odd level independence relations follow from the even ones).*

Proof. We proceed along the lines of the proofs of Lemmas 4.5, 4.7 and Theorem 4.1, from the previous section. If F is a subset in a von Neumann algebra and v a partial isometry with $vv^* = v^*v$, then we denote

$$F_{v,n}^k = \{\Pi_{j=1}^k v^{i_j} x_j \mid x_j \in F, 1 \leq |i_j| \leq n, 1 \leq j \leq k\}.$$

We first prove the following:

Fact. Let M be a finite von Neumann algebra and $A \subset M$ a MASA. Given any finite set $F \subset M \ominus A$, with $\|x\| \leq 1$, $\forall x \in F$, any $n \geq 1$ and any $\delta > 0$, there exists a Haar unitary $v \in A$ such that $|\tau(x)| \leq \delta$, $\forall x \in \bigcup_{k=1}^3 F_{v,n}^k$.

To prove this, denote by $\mathscr{W} = \{v \in A \mid vv^* \in \mathcal{P}(A), |\tau(x)| \leq \delta\tau(v^*v), \forall x \in \bigcup_{k=1}^3 F_{v,n}^k, \tau(v^m) = 0, \forall m \neq 0\}$. Endow \mathscr{W} with the order \leq in which $w_1 \leq w_2$ iff $w_1 = w_2 w_1^* w_1$. (\mathscr{W}, \leq) is then clearly inductively ordered. Let v be a maximal element in \mathscr{W} . Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$. If w is a partial isometry in Ap and $u = v + w$, then by using that $u^{i_j} = v^{i_j} + w^{i_j}$ and expanding

$x = u^{i_1}x_1u^{i_2}x_2\dots u^{i_k}x_k \in F_{u,n}^k$, $k = 1, 2, 3$, as a binomial product, we have $\tau(x) = \tau(\prod_{j=1}^k v^{i_j}x_j) + \Sigma\tau(\dots x_{j-1}w^{i_j}x_j\dots)$, and thus

$$(1) \quad |\tau(x)| \leq |\tau(\prod_{j=1}^k v^{i_j}x_j)| + \Sigma|\tau(\dots x_{j-1}w^{i_j}x_j\dots)|,$$

where the sum is taken over all terms that have at least one occurrence of w^{i_j} . Since $v \in \mathcal{W}$, we have $|\tau(\prod_{j=1}^k v^{i_j}x_j)| \leq \delta\tau(vv^*)$. We will prove that we can choose $w \neq 0$ so that the summation on the right hand side of (1) is majorized by $\delta\tau(ww^*)$, giving $|\tau(x)| \leq \delta\tau(vv^*) + \delta\tau(ww^*) = \delta\tau(uu^*)$. This will contradict the maximality of v , thus showing that $vv^* = 1$, i.e v a Haar unitary in A . We construct w by first making an appropriate choice for its support projection $q = ww^*$, then choosing w as an appropriate Haar unitary in Aq .

In order to estimate the summation $\Sigma|\tau(\dots x_{j-1}w^{i_j}x_j\dots)|$ in (1), note the following: in case $k = 1$ the sum has just one member, being of the form $|\tau(w^jy)|$, for some $1 \leq |j| \leq n$, $y \in X$; in case $k = 2$, the sum has three terms, being of the form

$$(2) \quad |\tau(w^{j_1}x_1v^{j_2}x_2)| + |\tau(v^{j_1}x_1w^{j_2}x_2)| + |\tau(w^{j_1}x_1w^{j_2}x_2)|;$$

in case $k = 3$, the sum has seven terms, being of the form

$$(3) \quad |\tau(w^{j_1}x_1v^{j_2}x_2v^{j_3}x_3)| + |\tau(v^{j_1}x_1w^{j_2}x_2v^{j_3}x_3)| + |\tau(v^{j_1}x_1v^{j_2}x_2w^{j_3}x_3)| \\ + |\tau(w^{j_1}x_1w^{j_2}x_2v^{j_3}x_3)| + |\tau(v^{j_1}x_1w^{j_2}x_2w^{j_3}x_3)| + |\tau(w^{j_1}x_1v^{j_2}x_2w^{j_3}x_3)| \\ + |\tau(w^{j_1}x_1w^{j_2}x_2w^{j_3}x_3)|.$$

Now note that for each summand for which we have 2 or 3 appearances of non-zero powers of w in the above sums (one term for $k = 2$ and four terms for $k = 3$), such appearances must be consecutive, i.e. they will be of the form $|\tau(\dots w^i y w^j \dots)|$, for some $i, j \neq 0$, $y \in F \subset M \ominus A$ (for one of the terms, one uses the equality $\tau(w^{j_1}x_1v^{j_2}x_2w^{j_3}x_3) = \tau(x_1v^{j_2}x_2w^{j_3}x_3w^{j_1})$). If $q = ww^*$, then for each one of these terms we have $|\tau(\dots w^i y w^j \dots)| \leq \|qyq\|_1$. By (2.1 in [P1], or A.1 in [P7]), or by using 3.6(d') in this paper, applied to the MASA $Ap \subset pMp$ and the set $pFp \subset pMp \ominus Ap$, one can choose the non-zero projection $q \in Ap$ such that $\|qyq\|_1 \leq 2^{-3}\delta\tau(q)$, $\forall y \in pFp$. It thus follows that the sum of terms having two or more appearances of powers of w are majorized by $2^{-1}\delta\tau(q)$ (because there are 2 such terms when $k = 2$ and 4 when $k = 3$).

All remaining terms and the case $k = 1$ have just one occurrence of w^j , $j \neq 0$, i.e are of the form $|\tau(y_1w^jy_2)| = |\tau(w^jE_A(qy_2y_1q))|$, for some $y_1, y_2 \in M$, $1 \leq |j| \leq n$. There are k many such terms for each $k = 1, 2, 3$. Let's denote by Y_0 the set of

all y_1, y_2 which appear this way, and note that this is a finite set in M . Thus $Y = E_A(qY_0 \cdot Y_0q)$ is finite as well. It is sufficient to find now a Haar unitary in Aq such that $|\tau(w^j y)| \leq 2^{-8}\delta\tau(q)$, $\forall y \in Y$, $1 \leq |j| \leq n$, because then the sum of the k terms will be majorized by $2^{-1}\delta\tau(q)$ which added up to the quantity $2^{-1}\delta\tau(q)$ that majorizes the terms with at least 2 occurrences of powers of w gives that $\forall x \in \cup_{k=1}^3 F_{u,n}^3$, we have $|\tau(x)| \leq \delta\tau(uu^*)$. Since Aq is diffuse, it contains a separable diffuse subalgebra $A_0 \subset Aq$, which is isomorphic to $L^\infty(\mathbb{T})$ with the Lebesgue measure corresponding to $\tau(q)^{-1}\tau|_{A_0}$. Let then $w_0 \in A_0$ be a Haar unitary generating A_0 . Since w_0^m tends to 0 in the weak operator topology and $Y \subset A$ is a finite set, there exists $n_0 \geq n$ such that $|\tau(w_0^m y)| \leq 2^{-4}\delta\tau(q)$, for all $y \in Y$ and $|m| \geq n_0$. But then $w = w_0^{n_0}$ is still a Haar unitary and it satisfies all the required conditions.

This ends the proof of the *Fact*.

With this in hand, we proceed as follows: Let $X_0 \subset Q$ be $\|\cdot\|_2$ -dense countable subset and denote by $X = \{y/\|y\| \mid y = x - E_{\mathbf{A}}(x), x \in X_0 \setminus \mathbf{A}\}$. Note that X is a countable subset of $\mathbf{M} \ominus \mathbf{A}$ and each element in X has operator norm equal to 1. Write X as a sequence $\{x_n\}_n$. For each n we now apply Step 1 to the set $F_n = \{x_1, \dots, x_n\}$ and $\delta = 1/n$, to get a Haar unitary $v_n \in \mathbf{A}$ such that

$$(4) \quad |\tau(\Pi_{j=1}^k v_n^{i_j} x_{t_j})| < 1/n, \forall |i_j|, t_j \in \{1, \dots, n\}, k = 1, 2, 3.$$

Let $x_n = (x_{n,m})_m$, $v_n = (v_{n,m})_m$ be representations of the x_n 's and v_n 's with $x_{n,m} \in M_m$, $\|x_{n,m}\| \leq 1$, $v_{n,m} \in \mathcal{U}(A_m)$. Thus, (4) and the fact that v_n are Haar unitaries, translates into

$$(5) \quad \lim_{m \rightarrow \omega} \tau(v_{n,m}^k) = 0, \forall k \neq 0; \lim_{m \rightarrow \omega} |\tau(\Pi_{j=1}^k v_{n,m}^{i_j} x_{t_j,m})| < 1/n, \forall |i_j|, t_j \in \{1, \dots, n\}.$$

From (5), it follows that for each n , there exists a neighborhood \mathcal{V}_n of ω such that for all $m \in \mathcal{V}_n$ we have

$$(6) \quad |\tau(v_{n,m}^k)| < 1/n, 1 \leq |k| \leq n; |\tau(\Pi_{j=1}^k v_{n,m}^{i_j} x_{t_j,m})| < 1/n, \forall |i_j|, t_j \in \{1, \dots, n\}.$$

Finally, let $u = (u_m)_m \in \mathbf{A}$ be defined by $u_m = v_{n,m}$, for all $m \in \mathcal{V}_n \setminus \mathcal{V}_{n-1}$. It is then immediate to check that u is a Haar unitary element in \mathbf{A} and that the von Neumann algebra B_0 it generates satisfies the required 3-independence conditions. \square

5.4. More on the incremental patching method. The technique used in Section 4 and in 5.4.1 above, of building Haar unitaries u that are n -independent with respect to certain sets, by “patching” together infinitesimal pieces of u , was

first considered (2.1 on [P5]), to prove that given any countable set X in a finite von Neumann algebra M and any diffuse abelian von Neumann subalgebra $A \subset M$, there exists a Haar unitary $u \in A^\omega$ such that any word that alternates letters from X and positive powers of u , has 0-trace. It was then used systematically in [P8] to prove that given any sequence of inclusions of finite von Neumann algebras $N_n \subset M_n$ with the property that for each n there exists an increasing sequence of intermediate algebras $N_n \subset N_n^k \nearrow M_n$ such that $N_n' \cap N_n^k$ is atomic $\forall k, n$, then given any countable set $Y \subset \Pi_\omega M_n \ominus (\Pi_\omega N_n' \cap \Pi_\omega M_n)$, there exists a Haar unitary $u \in \Pi_\omega N_n$ such that $\{u^n \mid n \in \mathbb{Z}\}$ and Y are *free independent relative to* $(\Pi_\omega N_n)' \cap \Pi_\omega M_n = \Pi_\omega(N_n' \cap M_n)$, i.e. any alternating word with letters in Y and $\{u^n \mid n \neq 0\}$ has 0 expectation on $(\Pi_\omega N_n)' \cap \Pi_\omega M_n$. As a consequence, the following result is obtained (2.1 in [P8]):

5.4.1. Theorem [P8]. *Let $\{N_n \subset M_n\}_n$ be a sequence of inclusions of type II_1 von Neumann algebras. Assume that for each n there exists an increasing sequence of intermediate von Neumann subalgebras $N_n \subset N_n^k \nearrow M_n$ such that $N_n' \cap N_n^k$ is atomic $\forall k$. Let P_0, P_1 be separable von Neumann subalgebras of $\Pi_\omega M_n$ with a common subalgebra $B \subset P_0, P_1$, such that $E_{\Pi_\omega N_n' \cap \Pi_\omega M_n}(P_i) = B$ for $i = 0, 1$, i.e., for each $i = 0, 1$, the algebras $\Pi_\omega N_n' \cap \Pi_\omega M_n$ and P_i make a commuting square, with intersection $(\Pi_\omega N_n' \cap \Pi_\omega M_n) \cap P_i = B$. Then there exists a unitary element $v \in \Pi_\omega N_n$ such that $P_0 \vee vP_1v^* = P_0 *_B vP_1v^*$, i.e. the algebras P_0 and vP_1v^* generate an amalgamated free product over B .*

As already pointed out in (2.2. of [P8]; 4.3 in [P10]), the above theorem implies paving results (equivalently, relative Dixmier property) for ultrapowers of inclusions of factors with trivial relative commutant.

But in fact, 5.4.1 had several other applications over the years: Thus, it played an important role in developing reconstruction methods in Jones theory of subfactors in ([P6], [P9], [P11]; see also the remarks in Section 0 and 2.5 in [P8]). Related to this, it led to the definition of amalgamated free products of inclusions of finite von Neumann algebras in [P6] (based also on the definition of reduced- C^* amalgamated free product over a conditional expectation in [V]). It was also used to prove key technical results in ([IPeP], [Va]) and to show that the free product of standard invariants of subfactors defined in ([BiJ]) can be realized in the hyperfinite II_1 factor R (see A.3 in [IPeP] and [Va]). On the other hand, Theorem 5.4.1 provides embedability results for amalgamated free products of algebras. For instance, by applying 5.4.1 to the case $N_n = M_n$ are all equal to the same factor M (e.g. $M = R$), one gets that if P_0, P_1 can be embedded into M^ω , then so can $P_0 *_B P_1$. More generally, 5.4.1 shows that if P_0, P_1 are embeddable into M^ω and $B \subset P_0, P_1$ is a common atomic subalgebra then $P_0 *_B P_1$ is embeddable into M^ω as well.

5.5. Free actions on A^ω have Bernoulli quotients. A concrete example of a subalgebra $B_0 \subset A^\omega$ satisfying the 3-independence property in Theorem 5.3.1 (Thus 3.9 as well) is the following: Let $\Gamma \curvearrowright X$ be an ergodic (but not necessarily free) measure preserving action of a discrete group Γ on a probability space (X, μ) and $\Gamma \curvearrowright Y = [0, 1]^\Gamma$ be the Bernoulli Γ -action with diffuse base. Let $A = L^\infty(X) \otimes L^\infty(Y)$ with $\Gamma \curvearrowright A$ the product action. Let $M = A \rtimes \Gamma$. If we take $B_1 = L^\infty(X)$, $Q = B_1 \rtimes \Gamma \subset M$ and $B_0 = 1 \otimes L^\infty([0, 1]) \otimes 1 \subset L^\infty(Y)$, the base of the Bernoulli action, viewed as a tensor component of the infinite tensor product $L^\infty(Y) = \otimes_{g \in \Gamma} (L^\infty([0, 1]))_g$, then it is easy to see that B_0 is 3-independent with respect to $Q \ominus B_1$. In fact, even if we take Q to be M itself, with $B_1 = A$, we can take inside A^ω the algebra B_0 to be defined as follows: let $g_n \in \Gamma$ be a sequence that tends to ∞ in Γ and $B_0 = \{(g_n(x))_n \mid x \in L^\infty([0, 1])\} \subset A^\omega$. Then B_0 is 3-independent to $M \ominus A$. One should notice that B_0 this way constructed is the base of a Bernoulli action on $A_1 = \vee_{g \in \Gamma} g(B_0)$, with A_1 τ independent to A and the von Neumann algebra generated by Γ, A, A_1 is isomorphic to $A \otimes A_1 \rtimes \Gamma$.

This construction can actually be recovered “asymptotically” inside any group measure space von Neumann algebra, as we will show next. The proof of this result is another application of the method of controlling moments through “incremental patching”.

5.5.1. Theorem. *Let $A_n \subset M_n$ be a sequence of MASAs in finite factors with $\dim M_n \rightarrow \infty$ and denote $\mathbf{A} = \Pi_\omega A_n \subset \Pi_\omega M_n = \mathbf{M}$. Assume $\Gamma \subset \mathcal{N}_\mathbf{M}(\mathbf{A})$ is a countable group of unitaries acting freely on \mathbf{A} . Given any separable abelian von Neumann subalgebra $B_1 \subset \mathbf{A}$, there exists a separable diffuse abelian subalgebra $A \subset \mathbf{A}$ such that: A, B_1 are τ -independent, Γ normalizes A , and the action of Γ on A is a Bernoulli Γ -action with diffuse base.*

Proof. Let $\{u_g \mid g \in \Gamma\}$ be the unitaries in Γ . We will construct a Haar unitary v in \mathbf{A} such that B_1 and $u_g\{v^n \mid n \in \mathbb{Z}\}u_g^*, g \in \Gamma$, are all mutually independent, i.e. given any finite set of distinct group elements $g_1, \dots, g_k \in \Gamma$, any non-zero integers n_1, \dots, n_k , and any $b \in B_1$, we have $\tau(b \prod_{j=1}^k u_{g_j} v^{n_j} u_{g_j}^*) = 0$. Let $\Gamma = \{g_n\}_n$ be an enumeration of $\Gamma \setminus \{1\}$ and $\{b_n\}_n$ a $\|\cdot\|_2$ -dense subset of the unit ball of B_1 . If v is a partial isometry in \mathbf{A} , then we denote by $F_{v,n}$ the set of all elements of the form $b_i \prod_{j=1}^k u_{g_j} v^{m_j} u_{g_j}^*$, where $1 \leq i \leq n$, $1 \leq k \leq n$, $|m_j| \leq n$.

We first prove the following:

Fact. Given any $n \geq 1$ and any $\delta > 0$, there exists a Haar unitary $v \in \mathbf{A}$ such that $|\tau(x)| \leq \delta, \forall x \in F_{v,n}$.

To prove this, denote by $\mathscr{W} = \{v \in A \mid vv^* \in \mathcal{P}(A), |\tau(x)| \leq \delta \tau(v^*v), \forall x \in F_{v,n}, \tau(v^m) = 0, \forall m \neq 0\}$. Endow \mathscr{W} with the order \leq in which $w_1 \leq w_2$ iff

$w_1 = w_2 w_1^* w_1$. (\mathcal{W}, \leq) is then clearly inductively ordered. Let v be a maximal element in \mathcal{W} . Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$. If $w \in \mathbf{A}p$ is a partial isometry satisfying $\tau(w^k) = 0$, $\forall k \neq 0$, and we denote $u = v + w$, then we have

$$(1) \quad b_j \Pi_{j=1}^k u_{g_j} u^{m_j} u_{g_j}^* = b_j \Pi_{j=1}^k u_{g_j} v^{m_j} u_{g_j}^* + \Sigma b_j \Pi_{j=1}^k u_{g_j} z^{m_j} u_{g_j}^*$$

where $z \in \{v, w\}$ and the sum is taken over all possible choices for $z = v$ or $z = w$, with at least one time $z = w$ (thus $2^k - 1$ many terms in the summation). We thus get the estimate

$$(2) \quad |\tau(b_j \Pi_{j=1}^k u_{g_j} u^{m_j} u_{g_j}^*)| \leq |\tau(b_j \Pi_{j=1}^k u_{g_j} v^{m_j} u_{g_j}^*)| + \Sigma |\tau(b_j \Pi_{j=1}^k u_{g_j} z^{m_j} u_{g_j}^*)| \\ \leq \delta\tau(vv^*) + \Sigma' |\tau(b_j \Pi_{j=1}^k u_{g_j} z^{m_j} u_{g_j}^*)| + \Sigma'' |\tau(b_j \Pi_{j=1}^k u_{g_j} z^{m_j} u_{g_j}^*)|$$

where the summation Σ' contains the terms with just one occurrence of $z = w$ and Σ'' is the summation of the terms that have at least 2 occurrences of $z = w$. Since \mathbf{A} is abelian, the terms $u_{g_j} z^{m_j} u_{g_j}^*$ in a product can be permuted arbitrarily. Thus, in each summand of Σ'' we can bring two of the occurrences of w so that to be adjacent, thus of the form $y_1 u_{g_j} w^{m_j} u_{g_j}^* u_{g_i} w^{m_i} u_{g_i}^* y_2$. Since $g_i \neq g_j$ for all $i \neq j$ and the group Γ acts freely on \mathbf{A} , it follows that there exists a non-zero $q \in \mathcal{P}(\mathbf{A}p)$ such that $qu_{g_j}^* u_{g_i} q = 0$ for all $1 \leq i \neq j \leq n$. This shows that Σ'' is 0 for any choice of w having support q satisfying this condition. We now choose w exactly like in the proof of 5.4.1, to make $\Sigma' \leq \delta\tau(w w^*)$, altogether showing that $|\tau(b_j \Pi_{j=1}^k u_{g_j} u^{m_j} u_{g_j}^*)| \leq \delta\tau(vv^*) + \delta\tau(w w^*) = \delta\tau(uu^*)$. This ends the proof of the *Fact*.

Using this *Fact*, one then proceeds exactly as in the last part of the proof of 5.4.1 to obtain a Haar unitary v in \mathbf{A} such that B_1 and $u_g \{v^n \mid n \in \mathbb{Z}\} u_g^*$, $g \in \Gamma$, are all mutually independent. □

5.6. Exact paving size for ultraproducts of singular MASAs. The order of magnitude of the paving size in Corollary 4.3 should be ε^{-2} , for any x , not only for $x = v$ unitary element with $E_{\mathbf{A}}(v) = 0$ and projections that expect on scalars (i.e., the cases covered by Cor 4.2). We pose here two questions which, if answered in the affirmative, would imply this fact:

(a) Can any x with $E_{\mathbf{A}}(x) = 0$ and norm $\leq 1/2$ (or of norm $\leq c$ for some other universal constant $c > 0$) be written as a convex combination of unitaries having 0-expectation on \mathbf{A} ? If so, then 4.2.1° would imply that $n(x, \varepsilon)$ is majorized by a constant multiple of ε^{-2} for any $x \in M^\omega$.

(b) Is it true that if M is a II_1 factor and $x = x^* \in M \ominus \mathbb{C}$, $u \in \mathcal{U}(M)$ a Haar unitary, such that $\tau(u^{i_1} x u^{i_2} x \dots) = 0$, for any alternating word with $i_j \neq 0$, then $\|\sum_{i=1}^n u^i x u^{-i}\|$ has order of magnitude \sqrt{n} ? Again, if this would hold true in this generality, then we would not need Proposition 2.3 at the end of the proof of Theorem 4.3, the result following directly from 4.1(a), with the estimate ε^{-2} for the order of magnitude of the paving size.

5.7. Final comments on Kadison-Singer. While we have not been able to settle the classic Kadison-Singer problem in its equivalent formulations of Theorem 2.2, i.e., by proving that one can pave all elements in R^ω (resp. in $\mathbf{M} = \Pi_\omega M_{n \times n}(\mathbb{C})$) over its MASA D^ω (resp. $\mathbf{D} = \Pi_\omega D_n$), we believe this is true and that the strategy we have developed here can be used to settle the problem. Of course, the resulting paving would be “non-free” in general. One should mention in this respect that a more careful “incremental patching” analysis as in Section 4, shows that any $x \in R^\omega \ominus \mathcal{N}(D^\omega)''$ (resp. $x \in \mathbf{M} \ominus \mathcal{N}(\mathbf{D})''$) can be paved. As for $x = x^* \in \mathcal{N}(D^\omega)'' \ominus D^\omega$ with $\|x\| \leq 1$, one can show the following: $\forall \varepsilon > 0$, $\exists p_i \in \mathcal{P}(D^\omega)$ finite partition, such that $y = \sum_i p_i x p_i$ satisfies $\|yq - vq\|_2 \geq (1 - \varepsilon)\|q\|_2$, for all $v \in \mathcal{N}(D^\omega)$, $q \in \mathcal{P}(D^\omega)$. In other words, one can make x “almost” perpendicular to $\mathcal{N}(D^\omega)$. Similarly for $\mathbf{D} \subset \mathbf{M}$. From this point on, the incremental control of the moments should be done so that for some fixed, universal $1 > c > 0$, one has $\tau(y^{2n}) \leq c^{2n}$. We do believe this can be done, and that in fact the following more general conjecture holds true (which by 2.2 solves the classic KS as well):

5.7.1. Conjecture: Given any sequence of MASAs in finite factors, $A_n \subset M_n$, the ultraproduct inclusion $\Pi_\omega A_n \subset \Pi_\omega M_n$ has the Kadison-Singer (equivalently, the paving) property.

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MATH.DEPT., UCLA, LOS ANGELES, CA 90095-1555

E-mail address: popa@math.ucla.edu